Decoding codes with locality

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Outline

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1. Motivation

2. Local recovery

3. Regeneration

4. Local correction

Applications of codes

Main application of codes: **communication** in presence of noise.

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Algebraic codes are now also used for **data storage**:

- on discs, e.g. Reed–Solomon codes in CDs, DVDs, hard disk drives,...
- for labelling or redirecting: bar codes, QR codes,...
- for massive data storage in **distributed storage systems** (DSS),
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- "easy" to download data from servers:
  - low bandwidth
  - few servers to be contacted
  - multiple/balanced access to servers
- resilience to server failures, delay, etc.
- security against eavesdroppers, corruption, etc.

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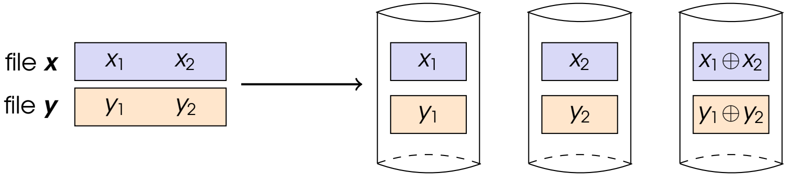
Codes are used to distribute files on the servers, and ensure these features.



A first simple storage model

Context:

- n servers $\mathcal{S}_1, \dots, \mathcal{S}_n$,
- each file is split into information symbols $a_1, \dots, a_k$,
- we assume $a_i \in \mathbb{F}_q$ (up to file striping).

file $\boxed{a_1 \ a_2 \ \cdots \ a_k}$

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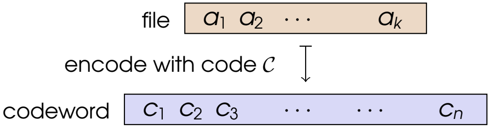

Consider a code

$$\mathcal{C} : \mathbb{F}_q^k \to \mathbb{F}_q^n$$

Encode the file $\boldsymbol{a} = (a_1, \dots, a_k)$
into a codeword $\boldsymbol{c} = (c_1, \dots, c_n) \in \mathcal{C}$.

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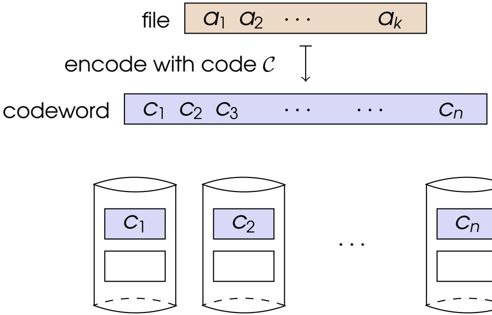

Same context as before, but codes are **array codes**:

$$\mathcal{C} : \mathbb{F}_q^k \to \mathbb{F}_q^{\alpha \times n}$$

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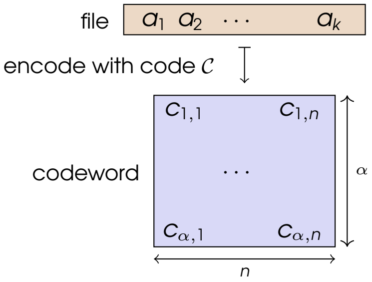

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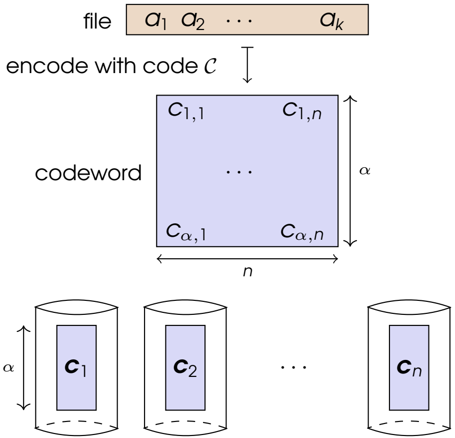

Relevant quantities

Low redundancy: store files with low redundancy.

Extractability: download files with few servers access.

Recovery/regeneration: if some server $\mathcal{S}_i$ fails, recover/regenerate the data stored on it:

- with few servers access: see **locally recoverable codes**
- with low bandwidth: see **regenerating codes**

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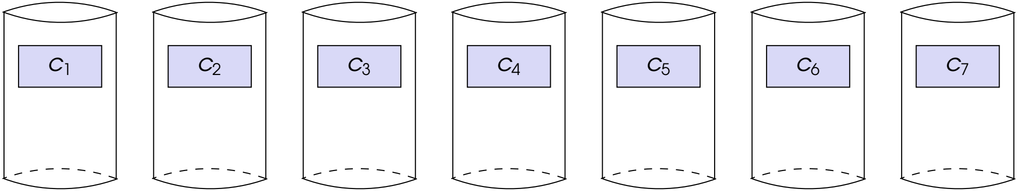
Illustration for locality



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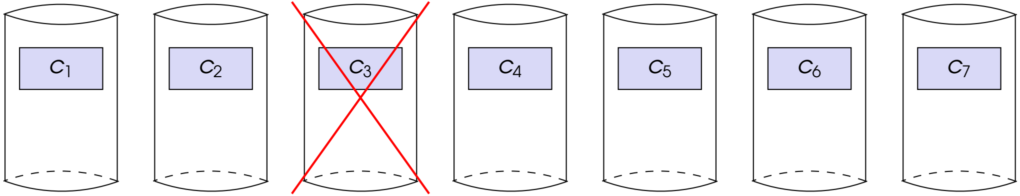

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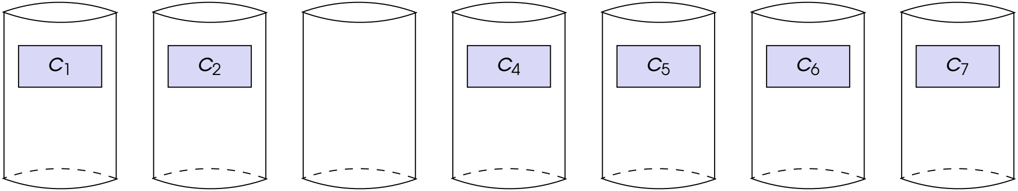

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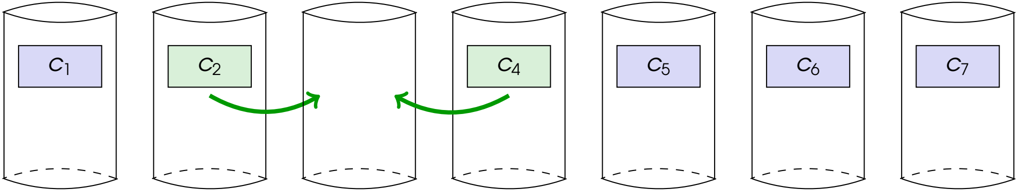

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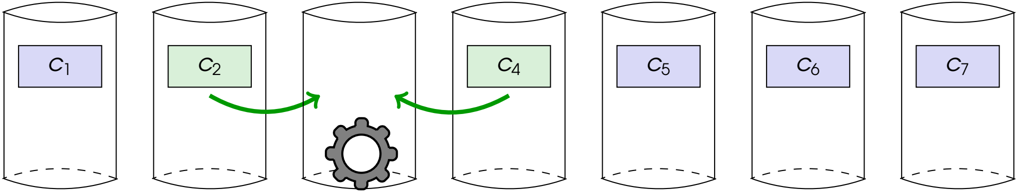

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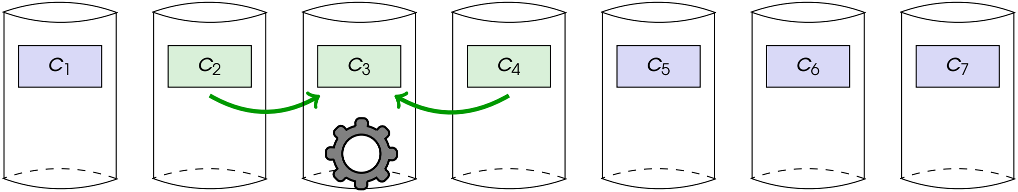

Symbol and code locality

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Let $\mathcal{C} \subseteq \mathbb{F}_q^n$ be a code of dimension k.

Def. A coordinate $i \in [1, n]$ has **locality** $r \geq 1$ in the code $\mathcal{C}$, if there exists a subset $\mathcal{R}_i \subset [1, n] \setminus \{i\}$ of size at most r such that:

$$\dim \mathcal{C}_{|\mathcal{R}_i} = \dim \mathcal{C}_{|\mathcal{R}_i \cup \{i\}}$$

Subset $\mathcal{R}_i$ is called a **helper set** for coordinate i.

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Def. The code $\mathcal{C}$ is r**-locally recoverable** (r-LR) if each coordinate $i \in [1, n]$ has locality r in $\mathcal{C}$.

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Locality $r = k$. Choose I such that $i \notin I$, and compute c_i from $\mathcal{C}_{|I}$.

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Prop. Let $\mathcal{C} \subseteq \mathbb{F}_q^n$ be an $[n, k]_q$-LRC with locality r. Then we have

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Remark. MDS codes attain the upper bound. **Worst locality** compared to their dimension.

One can design LRCs via their **parity-check matrix** $\boldsymbol{H} \in \mathbb{F}_q^{(n-k)\times n}$.

$$\mathcal{C} = \left\{ \boldsymbol{c} \in \mathbb{F}_q^n \mid \boldsymbol{H}\boldsymbol{c}^\top = \boldsymbol{0} \right\}.$$

For instance, if

$$\boldsymbol{H} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

then $\mathcal{C}$ has locality $r = 3$.

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Q. Do we have a bound depending on the minimum distance d (as for Singleton bound)?

A Singleton-like bound

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Prop. Let $\mathcal{C}$ be an $[n, k, d]_q$-LRC with locality r. Then:

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Then $\mathcal{C}' := \mathrm{Short}(\mathcal{C}, \mathcal{R})$ is a code of

- length $n' \geq n - (r + 1)$;
- dimension $k' \geq k - r$ since $\dim \mathcal{C}_{|\mathcal{R}_i} = \dim \mathcal{C}_{|\mathcal{R}_i \cup \{i\}}$;
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Step 2: iterate this shortening process as much as possible and apply the classical Singleton bound ($d \le n - k + 1$) on the shortened code.

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Q. Do there exist optimal codes w.r.t. this "Singleton-like bound", for $k \neq r$?

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Idea. In **a well-chosen Reed–Solomon code** (large minimum distance), select a subspace of polynomials with local recovery properties.

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$$\forall x_1 \in \mathcal{X}_1,\ x_1^3 = 1; \qquad \forall x_2 \in \mathcal{X}_2,\ x_2^3 = 8; \qquad \forall x_3 \in \mathcal{X}_3,\ x_3^3 = 12.$$

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Evaluate over $\boldsymbol{x} = (1,3,9,2,6,5,4,12,10)$ the following polynomial fonctions:

$$\mathcal{F} := \langle 1, X, \quad X^3, X^4, \quad X^6, X^7 \rangle_{\mathbb{F}_q}\,.$$

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If $f(X) = \sum_i a_i X^i \in \mathcal{F}$, then for all $x \in \mathcal{A}_2$ (for instance)

$$f(x) = (a_0 + 8 \cdot a_3 + 8^2 a_6) \cdot 1 + (a_1 + 8 \cdot a_4 + 8^2 a_7) \cdot x = u_2 + v_2 \cdot x$$

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$$\mathcal{F} := \langle 1, X, \quad X^3, X^4, \quad X^6, X^7 \rangle_{\mathbb{F}_q}.$$

If $f(X) = \sum_i a_i X^i \in \mathcal{F}$, then for all $x \in \mathcal{A}_2$ (for instance)

$$f(x) = (a_0 + 8 \cdot a_3 + 8^2 a_6) \cdot 1 + (a_1 + 8 \cdot a_4 + 8^2 a_7) \cdot x = u_2 + v_2 \cdot x$$

Consequence. If we know $f(x')$ and $f(x'')$ for both other evaluation points x' and x'' of $\mathcal{A}_2$, we can solve a linear system to get u_2 and v_2, and then $f(x)$.

Tamo–Barg codes: definition

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Let $g \in \mathbb{F}_q[X]$ such that:

(i) $\deg(g) = r + 1$.

(ii) there exist pairwise distinct $y_1, \dots, y_s \in \mathbb{F}_q$ such that $\mathcal{X}_j := g^{-1}(\{y_j\})$ has size $r + 1$.

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Define the evaluation vector

$$\boldsymbol{x} = (\underbrace{x_{1,1}, \dots, x_{1,r+1}}_{\mathcal{X}_1}, x_{2,1}, \dots, \dots, x_{s,1}, \dots, x_{s,r+1}) \in \mathbb{F}_q^{s(r+1)}$$

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Define, for $0 \le \ell \le s$,

$$\mathcal{F}_\ell := \langle X^i g(X)^j \mid 0 \le i \le r-1, 0 \le j \le \ell - 1 \rangle_{\mathbb{F}_q} \quad \subseteq \mathbb{F}_q[X]_{<(r+1)\ell-1}\,.$$

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Def. The corresponding Tamo–Barg code is:

$$\mathrm{TB}_{r,s,\ell}(\boldsymbol{x}) = \{\mathrm{ev}_{\boldsymbol{x}}(f) \mid f \in \mathcal{F}_\ell\} \subseteq \mathbb{F}_q^{s(r+1)}$$

Tamo–Barg codes: optimality

Thm. The Tamo–Barg code $\mathrm{TB}_{r,s,\ell}(\boldsymbol{x}) = \{\mathrm{ev}_{\boldsymbol{x}}(f) \mid f \in \mathcal{F}_\ell\}$ with $\mathcal{F}_\ell := \langle X^i g(X)^j \rangle_{0 \le i \le r-1, 0 \le j \le \ell-1}$ has the following parameters:

1. length $n = (r+1)s$;
2. dimension $k = r\ell$;
3. minimum distance $d = (r+1)(s-\ell) + 2$
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Corollary. Tamo–Barg codes are optimal w.r.t. the Singleton-like bound:

$$d = n - k + 2 - \left\lceil \frac{k}{r} \right\rceil .$$

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Explicit Tamo–Barg codes

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1. Let H be a **multiplicative** subgroup of $\mathbb{F}_q^\times$ of order $r+1$.
   The polynomial
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Question. Good constructions of LRCs for $q < n$?

We can **generalize** this idea with AG codes.

Towards a generalization of Tamo–Barg codes

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Let $g : \mathcal{X} \to \mathcal{Y}$ be a rational map. Assume $\mathbb{F}_q(\mathcal{X}) = \mathbb{F}_q(\mathcal{Y})(x)$ is separable, with

$$x^{r+1} + \sum_{i=0}^{r} b_i x^i = 0, \quad b_i \in \mathbb{F}_q(\mathcal{Y})$$

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Let $\mathcal{P} = \{P_1, \dots, P_s\} \subseteq \mathcal{Y}(\mathbb{F}_q)$ and D be an effective divisor of degree ℓ, with $\mathrm{supp}(D) \cap \mathcal{P} = \varnothing$.

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(similar as before)

Finally, assume that $b_i \in L(n_i D)$ for some $n_i \ge 1$, and denote by $f_1, \dots, f_m \in \mathbb{F}_q(\mathcal{Y})$ a basis of $L(D)$. Then define:

$$\mathcal{F} := \langle x^i f_j \rangle_{0 \le i \le r-1, 1 \le j \le m} \subset \mathbb{F}_q(\mathcal{X})$$

(before: $\mathcal{F} = \{x^i g(x)^j \mid \dots\}$)

Locally Recoverable Codes on Algebraic Curves. A. Barg, I. Tamo, S. Vladut. IEEE TIT. **2017**.

Def. According to previous notation, we define a Barg–Tamo–Vladut code as:

$$\mathrm{BTV} = \{\mathrm{ev}_{\cup \mathcal{X}_j}(f) \mid f \in \mathcal{F}\} \subseteq \mathbb{F}_q^{s(r+1)}$$

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Thm. Code BTV has the following parameters:

- length $n = (r+1)s$,
- dimension $k = rm \geq t(\ell - g_{\mathcal{Y}} + 1)$,
- minimum distance $d \geq n - \ell(r+1) - (r-1)h$,
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where h is the degree of $x \in \mathbb{F}_q(\mathcal{X})$.

Barg–Tamo–Vladut's construction

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where h is the degree of $x \in \mathbb{F}_q(\mathcal{X})$.

Consequence. Explicit asymptotic families of codes exceeding a Gilbert–Varshamov-like bound for LRCs.

Other constructions

- "Pyramid codes":
  Pyramid codes: Flexible schemes to trade space for access efficiency in reliable data storage systems. C. Huang, M. Chen, and J. Li. Proc. IEEE ISNCA. **2007**.

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- Via sum-rank codes, linearized and subspace polynomials.
  Universal and dynamic locally repairable codes with maximal recoverability via sum-rank codes. U. Martínez-Penas and F. R. Kschischang. IEEE TIT. **2019**.
  Explicit maximally recoverable codes with locality. P. Gopalan, C. Huang, B. Jenkins, and S. Yekhanin. IEEE TIT. **2014**.

In pratice

LRCs have been tested and implemented:

- in *Windows Azure*: with an $(n = 16, k = 12, r = 6)$ and then an $(n = 18, k = 14, r = 7)$ LRC;
- in *Facebook storage system*, with an $(n = 16, k = 10, r = 5)$ LRC;
- in *Ceph* storage cluster

First extension: local recovery with correction

LRCs usually allow to recover 1 erased symbol (by accessing $\leq r$ other symbols).

Problem. Recover many erasures? Correct errors instead of erasures?

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First solution: improve the "local distance" to $\delta > 2$.

Def. A code $\mathcal{C} \subseteq \mathbb{F}_q^n$ is (r, δ)–locally recoverable if, for all $1 \leq i \leq n$, there exists a subset $\mathcal{R}_i \subset [1, n]$, such that $i \notin \mathcal{R}_i$, $|\mathcal{R}_i| \leq r$, and

- $\dim \mathcal{C}_{|\mathcal{R}_i} = \dim \mathcal{C}_{|\mathcal{R}_i \cup \{i\}}$,
- $\mathrm{d}_{\min}(\mathcal{C}_{|\mathcal{R}_i \cup \{i\}}) \geq \delta$.

A Singleton-like bound for (r, δ)-LRCs:

Prop. If $\mathcal{C}$ is an (r, δ)-LR code with parameters $[n, k, d]_q$, then

$$d \leq n - k + 1 - (\delta - 1)\left(\left\lceil \frac{k}{r} \right\rceil - 1\right) .$$

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Prop. If $\mathcal{C}$ is an (r, δ)-LR code with parameters $[n, k, d]_q$, then

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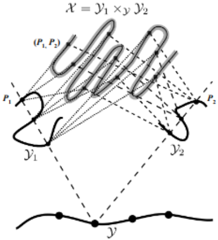
Optimal construction: generalization of Tamo–Barg codes.
→ modify the polynomial space into:

$$\mathcal{F} := \{x^i g(x)^j \mid 0 \leq i \leq r + 1 - \delta, 0 \leq j \leq \ell\} .$$

Second solution: many helper sets for the same coordinate.

Repair locality with multiple erasure tolerance. A. Wang and Z. Zhang. IEEE TIT. **2014**.

Def. A code $\mathcal{C}$ has locality r and **availability** t if, for all $i \in [1, n]$, the coordinate i admits t pairwise disjoint helper sets.



Remark. Nice geometric constructions with fiber products of curves.

Illustration and reference:

Locally recoverable codes with availability $t \geq 2$ from fiber products of curves. K. Haymaker, B. Malmskog, G. Matthews. Adv. Math. Comm.. **2018**.

Outline

1. Motivation

2. Local recovery

3. Regeneration

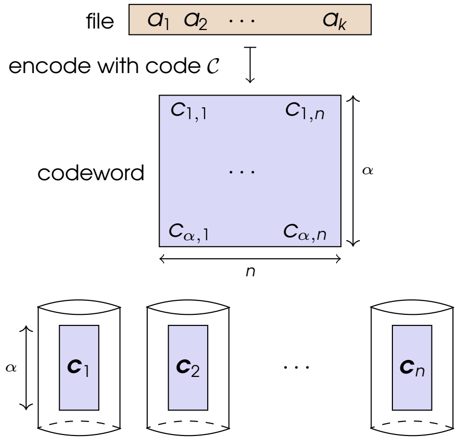
4. Local correction

Same context as before, but codes are **array codes**:

$$\mathcal{C} : \mathbb{F}_q^k \to \mathbb{F}_q^{\alpha \times n}$$

Encode the file $\boldsymbol{a} = (a_1, \dots, a_k)$
into a (matrix) codeword $\boldsymbol{C} = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_n) \in \mathcal{C}$.

Server $\mathcal{S}_i$ stores α symbols $\boldsymbol{c}_i \in \mathbb{F}_q^\alpha$.



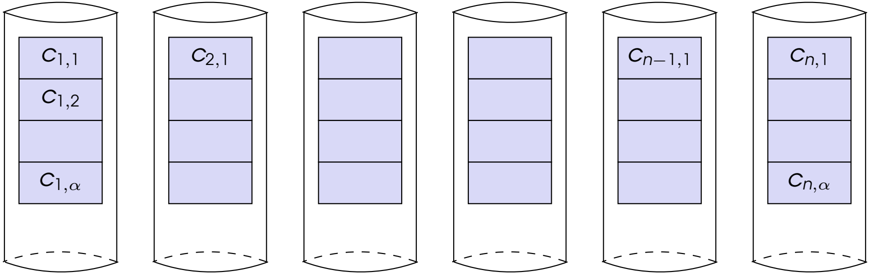
Def. An $(n, s, t, \alpha, \beta, k)$-**regenerating code** is a code $\mathcal{C} \subseteq \mathbb{F}_q^{\alpha \times n}$ of dimension k such that:

- each codeword $\boldsymbol{C} \in \mathcal{C}$ is fully determined by **any** s-subset of its columns *(data reconstruction property)*,
- each column $\boldsymbol{c}_i$ of $\boldsymbol{C}$ can be "regenerated", by downloading at most $\beta \leq \alpha$ symbols from **any** t-subset of columns *(regeneration property)*.

(rem: unusual notation; more usually $(n, k, d, \alpha, \beta, M)$-regenerating code)

Note. The regeneration condition enforces $t\beta \geq \alpha$.

Illustration for regeneration



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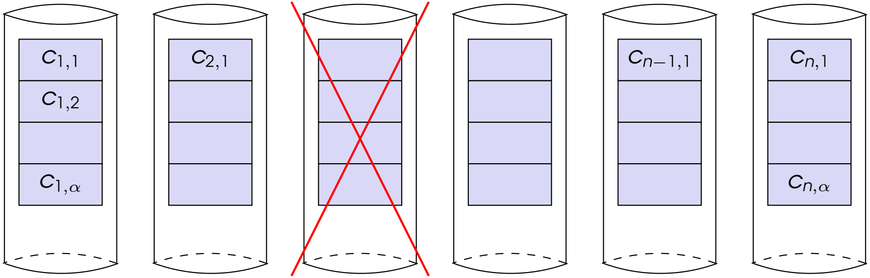

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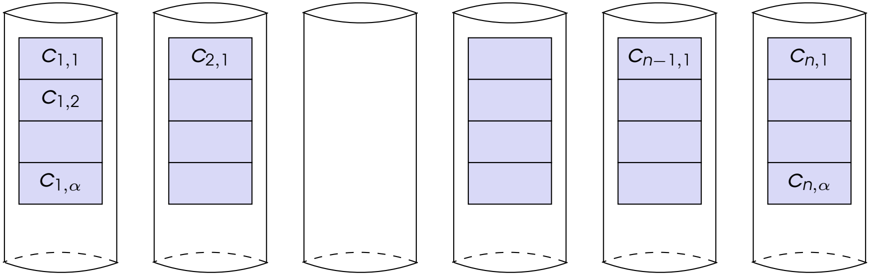

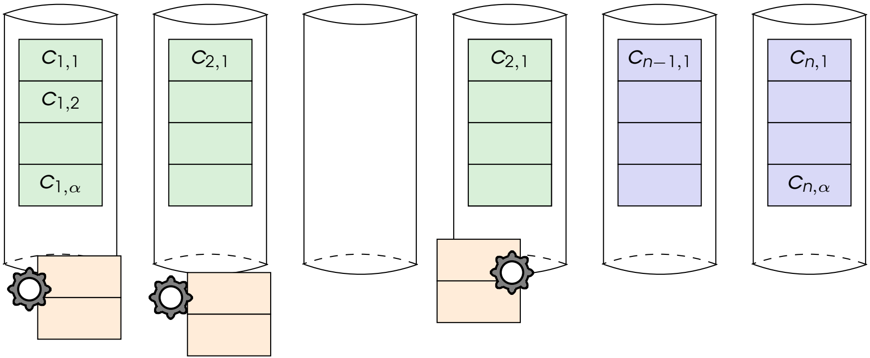

$t = 3$ servers are contacted for regeneration

$\beta = 2$ symbols are sent by each server for regeneration

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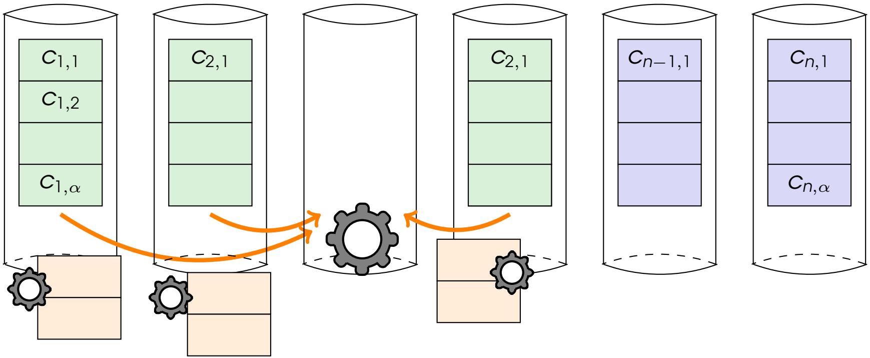

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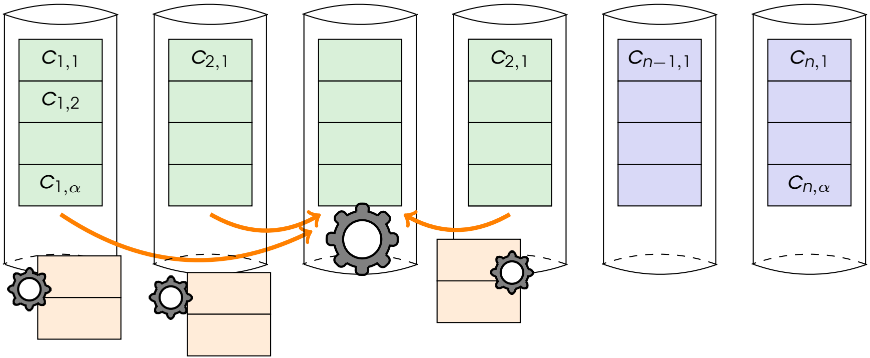

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Cut-set bound

Main bound (cut-set bound): Any $(n, s, t, \alpha, \beta, k)$-regenerating code satisfies

$$k \leq \sum_{i=0}^{s-1} \min(\alpha, (t-i)\beta)\,.$$

Rem. For fixed s and t, there are many optimal points (depending on α and β).

For instance, if $t\beta = \alpha$, the bound defines a **minimum-bandwidth repair (MBR) point**:

$$k = \left(st - \frac{s(s-1)}{2}\right)\beta\,.$$

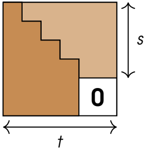
Deterministic regenerating codes for distributed storage. Y. Wu, A. G. Dimakis, and K. Ramchandran. Allerton conf. on control, computing, and communication. **2007**.

Product-matrix MBR codes

Optimal Exact-Regenerating Codes for Distributed Storage at the MSR and MBR Points via a Product-Matrix Construction. Rashmi, Shah, Kumar. IEEE-TIT. **2011**.

For the example, we set $\beta = 1$, hence $\alpha = t$.
Set also $k = \left(st - \frac{s(s-1)}{2}\right)\beta = st - \frac{s(s-1)}{2}$.

📄 *Optimal Exact-Regenerating Codes for Distributed Storage at the MSR and MBR Points via a Product-Matrix Construction*. Rashmi, Shah, Kumar. IEEE-TIT. **2011**.



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Set also $k = \left(st - \frac{s(s-1)}{2}\right)\beta = st - \frac{s(s-1)}{2}$.

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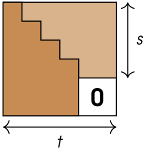
$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{S} & \boldsymbol{T}^\top \\ \boldsymbol{T} & \boldsymbol{0} \end{pmatrix}$$

Optimal Exact-Regenerating Codes for Distributed Storage at the MSR and MBR Points via a Product-Matrix Construction. Rashmi, Shah, Kumar. IEEE-TIT. **2011**.



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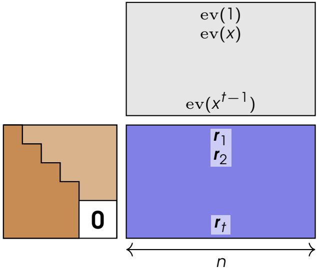
2. Let $\boldsymbol{G}$ be a $(t \times n)$ generator matrix for a Reed-Solomon code $\mathrm{RS}_q(t, n)$, echelonized in degree. **Codewords** are then:

$$\boldsymbol{C} = \boldsymbol{A}\boldsymbol{G} \in \mathbb{F}_q^{t \times n}.$$



MBR PM codes: data reconstruction

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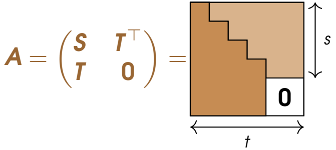

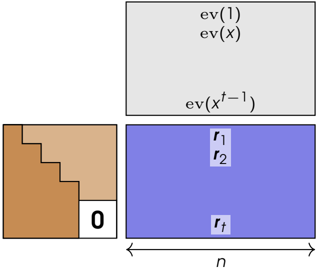

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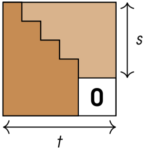
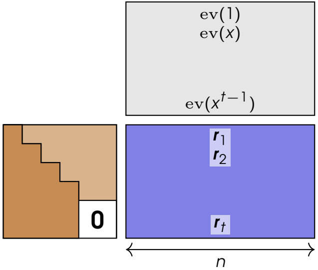
$A = \begin{pmatrix} S & T^\top \\ T & 0 \end{pmatrix} =$ 



Goal: recover $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{S} & \boldsymbol{T}^\top \\ \boldsymbol{T} & \boldsymbol{0} \end{pmatrix}$ from the codeword $\boldsymbol{C}$, with access to only s columns.

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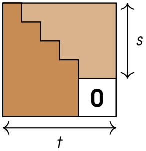
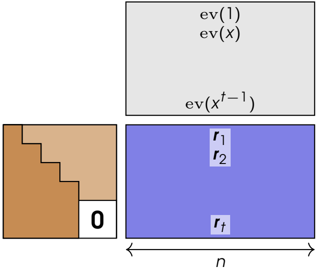
$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{S} & \boldsymbol{T}^\top \\ \boldsymbol{T} & \boldsymbol{0} \end{pmatrix} =$ 



MBR PM codes: data reconstruction

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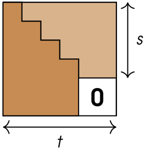
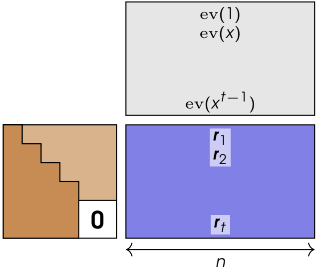
$\rightarrow$ we can recover $T_{1,i}, \dots, T_{s,i}$ with any s symbols of r_i.

$\rightarrow$ we can recover T with any s columns of C.

$A = \begin{pmatrix} S & T^\top \\ T & 0 \end{pmatrix} =$ 



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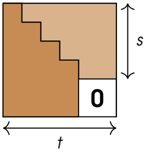
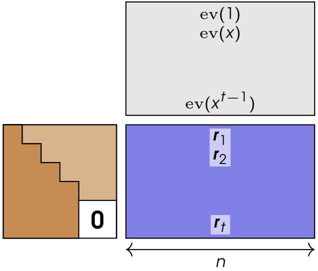
$\rightarrow$ we can recover $\boldsymbol{T}$ with any s columns of $\boldsymbol{C}$.

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{S} & \boldsymbol{T}^\top \\ \boldsymbol{T} & \boldsymbol{0} \end{pmatrix} =$$



Step 2: find $\boldsymbol{S}$.



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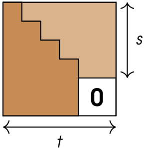
$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{S} & \boldsymbol{T}^\top \\ \boldsymbol{T} & \boldsymbol{0} \end{pmatrix} =$ 

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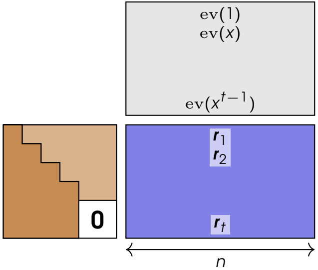

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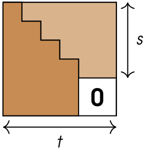
$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{S} & \boldsymbol{T}^\top \\ \boldsymbol{T} & \boldsymbol{0} \end{pmatrix} =$ 

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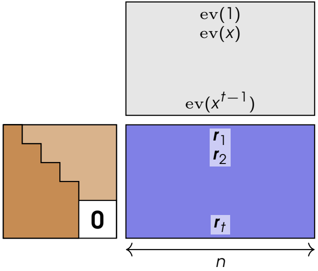
- $\rightarrow$ vector $\boldsymbol{r}_i - \boldsymbol{z}_i$ is an element $\mathrm{RS}_q(s, n)$.
- $\rightarrow$ we recover $S_{1,i}, \dots, S_{s,i}$, hence $\boldsymbol{S}$, as in Step 1, *i.e.* with any s columns of $\boldsymbol{C}$.



Goal: recover a column $\boldsymbol{c}_i$ of $\boldsymbol{C}$, by retrieving only $\beta = 1$ symbol from t servers.
Important: we allow servers to **compute** on data.

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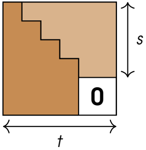

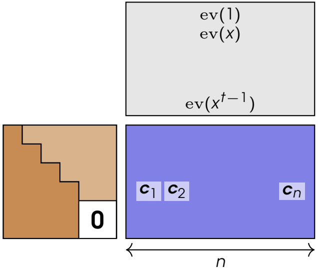

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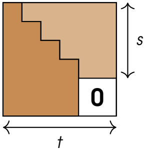
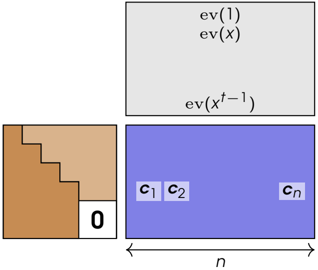
$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{S} & \boldsymbol{T}^\top \\ \boldsymbol{T} & \boldsymbol{0} \end{pmatrix} =$ 



MBR PM codes: regeneration

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$$\langle \boldsymbol{g}_i, \boldsymbol{c}_j \rangle = \langle \boldsymbol{g}_i, \boldsymbol{A}\boldsymbol{g}_j \rangle = \langle \boldsymbol{A}^\top \boldsymbol{g}_i, \boldsymbol{g}_j \rangle = \langle \boldsymbol{A}\boldsymbol{g}_i, \boldsymbol{g}_j \rangle = \langle \boldsymbol{c}_i, \boldsymbol{g}_j \rangle$$

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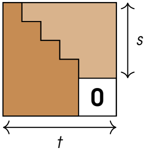

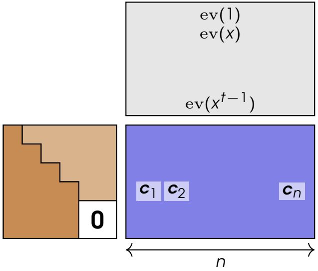

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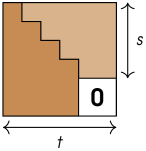

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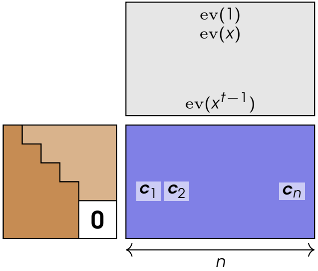
$$(y_1, \dots, y_t) =: \boldsymbol{y} = \boldsymbol{c}_i \boldsymbol{G}_{|J} \,.$$



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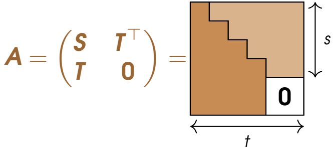

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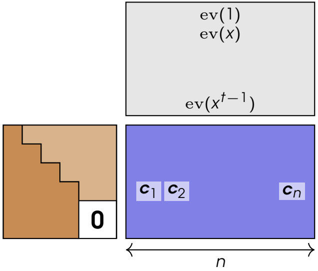
Step 2: after collection of $\boldsymbol{y} = (y_1, \dots, y_t)$, compute and recover

$$\boldsymbol{y}\boldsymbol{U} = \boldsymbol{c}_i \boldsymbol{G}_{|J} \boldsymbol{U} = \boldsymbol{c}_i\,.$$



Other works

- Quite surprisingly, **repair of Reed–Solomon codes over extension fields** $\mathbb{F}_q/\mathbb{F}_p$, using the $\mathbb{F}_p$-vector space structure and trace-dual bases.
  Repairing Reed–Solomon codes. V. Guruswami and M. Wootters. IEEE TIT. **2017**.
- Sequential recovery.
  Codes with locality for two erasures. N. Prakash, V. Lalitha, S. B. Balaji, and P. V. Kumar. IEEE TIT. **2019**.
- Hierarchical locality
  Codes with hierarchical locality. B. Sasidharan, G. K. Agarwal, and P. V. Kumar. IEEE ISIT. **2015**.
- Maximally Recoverable Codes
  On the maximally recoverable property for multi-protection group codes. M. Chen, C. Huang, and J. Li. ISIT. **2007**.
- A survey:
  Codes for Distributed Storage. Ramkumar, Balaji, Sasidharan, Vajha, Krishnan, Kumar. Foundations and Trends in Communication and Information Theory. **2022**.

Perspectives

Locality (and regeneration?) for codes defined w.r.t. **other metrics**:

- in **rank-metric**, attempted in 2017 by Kadhe, El Rouayheb, Duursma & Sprintson in *Codes with Locality in the Rank and Subspace Metrics* (framework to be discussed);
- in **weighted metrics**, *e.g.* for codes over the integers.

Locality for **dynamic** coded databases.

Locality and codes for **peer-to-peer storage** systems, with a large amount of disconnected nodes.

Locality with specific **storage topologies**, with nodes adjacency given by a graph.

Locality with **service rate** constaints.

Locality for **coded distributed computing**.

Outline

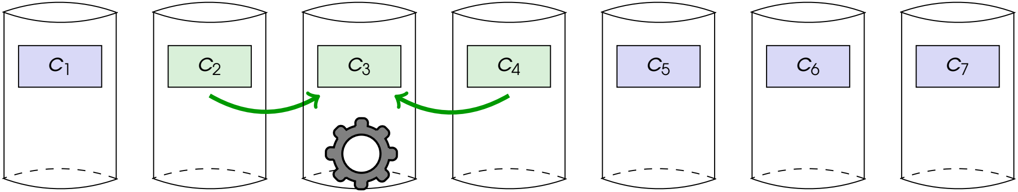
1. Motivation

2. Local recovery

3. Regeneration

4. Local correction

Illustration for locality



Reminder: availability (1)

Def. A code $\mathcal{C}$ has locality r and **availability** t if, for all $1 \leq i \leq n$, the coordinate i admits at least t pairwise disjoint helper sets.

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Tensor product of codes $\mathcal{A} \subseteq \mathbb{F}_q^m$ and $\mathcal{B} \in \mathbb{F}_q^n$:

$$\mathcal{A} \otimes \mathcal{B} = \left\{ \boldsymbol{c} \in \mathbb{F}_q^{m \times n} \;\middle|\; \begin{array}{l} \text{each column of } \boldsymbol{c} \text{ is in } \mathcal{A} \\ \text{each row of } \boldsymbol{c} \text{ is in } \mathcal{B} \end{array} \right\} \subseteq \mathbb{F}_q^{m \times n}$$

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There are $t = 2$ ways to retrieve $c_{i,j}$ efficiently:

1. select the i-th row of $\boldsymbol{c}$ (except $c_{i,j}$) and use a parity-check equation of $\mathcal{B}$:

$$c_{i,j} = \sum_{\ell \neq j} \lambda_\ell c_{i,\ell}$$

2. select the j-th column of $\boldsymbol{c}$ (except $c_{i,j}$) and use a parity-check equation of $\mathcal{A}$:

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Prop. If $\mathcal{A}$ and $\mathcal{B}$ are r-LRCs (without availability), then $\mathcal{A} \otimes \mathcal{B}$ is also r-LR, with availability $t = 2$.

Reminder: availability (2)

This can be extended to m-wise tensor products (or m-th tensor powers) of codes:

Prop. Let $\mathcal{C} \subseteq \mathbb{F}_q^n$ be an $[n, k, d]_q$-LRC with locality r. Then

$$\mathcal{C}^{\otimes t} := \mathcal{C} \otimes \cdots \otimes \mathcal{C}$$

is a code of length n^t, dimension k^t and minimum distance d^t. It is also an LRC with locality r and availability t.

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In a **worst case scenario**, $t + 1$ erasures lie on a coordinate i and on 1 coordinate of each of its t helper sets. Then,

- the symbol c_i cannot be decoded;
- we get no information about c_i.

This can be extended to m-wise tensor products (or m-th tensor powers) of codes:

Prop. Let $\mathcal{C} \subseteq \mathbb{F}_q^n$ be an $[n, k, d]_q$-LRC with locality r. Then

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For tensor codes, t is only $\log(\text{length}(\mathcal{C}))$...

We would like to retrieve **some information** about c_i, with few access to $\boldsymbol{c}$, and even with a large amount of errors/erasures.

Another paradigm

Usual decoding algorithms for a code $\mathcal{C} \subseteq \mathbb{F}_q^n$:

$$\boldsymbol{y} = \boldsymbol{c} + \boldsymbol{e} \in \mathbb{F}_q^n \longrightarrow \boldsymbol{c} \in \mathcal{C}$$

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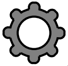

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New goal. Recover only **one** symbol c_i of a codeword $\boldsymbol{c} \in \mathcal{C}$, but in sublinear time $o(n)$, thus **probabilistically**.

1. How many errors can be handled?
2. Which probability of success?

Locally correctable codes

Consider a **locality** parameter $\ell \geq 2$, an **error ratio** $\tau \in (0, 1)$, a **soundness parameter** $\varepsilon < 1/2$.

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Def. A code $\mathcal{C} \in \mathbb{F}_q^n$ is said $(\ell, \tau, \varepsilon)$**-locally correctable** if there exists a (probabilistic) algorithm Cor such that, for all noisy codewords $\boldsymbol{y} = \boldsymbol{c} + \boldsymbol{e}$ with $\boldsymbol{c} \in \mathcal{C}$ and $|\boldsymbol{e}| \leq \tau n$, we have

1. *(soundness)* $\mathbb{P}(\mathrm{Cor}^{(\boldsymbol{y})}(i) \neq c_i) \leq \varepsilon$;
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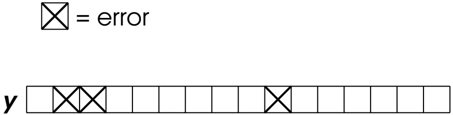
Rem. $\mathbb{P}$ is computed w.r.t. random coins of Cor, not over $\boldsymbol{e}$.



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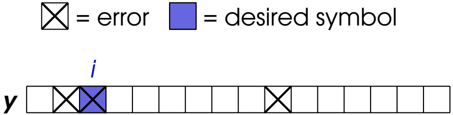

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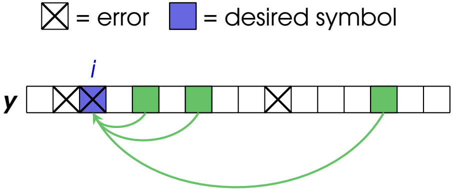

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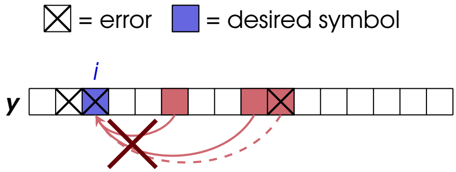

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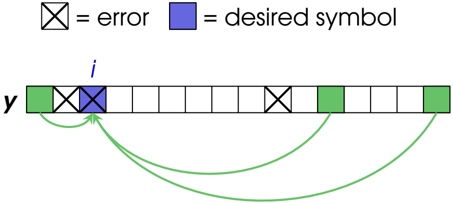

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- minimum distance is of secondary importance.

Hadamard code

Def. The **Hadamard code** consists in evaluation vectors of linear forms $\langle \boldsymbol{a} \mid \cdot \rangle$ (= homogeneous polynomials of degree 1) over all points $\boldsymbol{x} \in \mathbb{F}_q^k$:

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Thm. Hadamard code is $(\ell = 2, \tau, \varepsilon = 2\tau)$–locally correctable, for all $\tau < 1/2$.

(**Rem.** This is also an LRC with $r = 2$ and very high availability $t \simeq n/2$.)

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Q. Can we reach constant rate $k/n \in \Omega(1)$ with sublinear locality $\ell = o(n)$?

Def. Let $m \geq 1$ and $r \leq q-1$. The **Reed–Muller code** of order m and degree r over $\mathbb{F}_q$ is:

$$\mathrm{RM}_q(m,r) = \left\{ \mathrm{ev}_{\mathbb{F}_q^m}(f) := (f(\boldsymbol{x}))_{\boldsymbol{x} \in \mathbb{F}_q^m} \mid f \in \mathbb{F}_q[X_1, \dots, X_m],\ \deg f \leq r \right\}$$

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Remark. For $m=1$, we have a Reed–Solomon code.

Let $\boldsymbol{c} = \mathrm{ev}_{\mathbb{F}_q^m}(f) \in \mathrm{RM}_q(m, r)$ and $\phi : \mathbb{F}_q \to \mathbb{F}_q^m$ an affine map:

$$\phi(T) = (a_1 T + b_1, \dots, a_m T + b_m) \qquad \text{with } (a_1, \dots, a_m) \neq (0, \dots, 0)$$

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Let $L := \phi(\mathbb{F}_q)$ be the corresponding affine line of $\mathbb{F}_q^m$. Up to reordering the coordinates,

$$\mathrm{ev}_{\mathbb{F}_q}(f \circ \phi) = \mathrm{ev}(f)_{|L}$$

Local correction of Reed–Muller codes (1)

Local correcting algorithm for Reed–Muller codes $\mathrm{RM}_q(r, m)$ of degree $r \leq q - 2$.

Input:

- a coordinate $\boldsymbol{x} \in \mathbb{F}_q^m$
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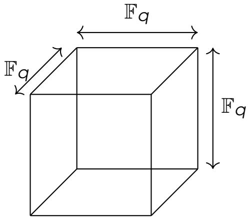

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1. draw at random an affine line $L \subset \mathbb{F}_q^m$ passing through $\boldsymbol{x}$,
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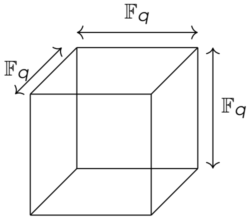

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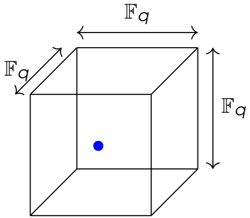

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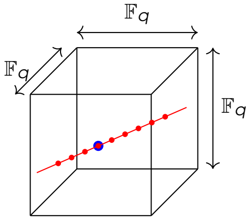

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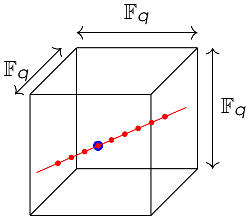

Remark: we require $r \le q-2$ so that the local Reed–Solomon code has distance ≥ 2.

Local correction of Reed–Muller codes (2)

Let

$$\tau = \frac{q-r-2}{2(q-1)} > 0$$

be such that the Reed-Solomon code $\mathrm{RS}_q(r+1)$ can correct 1 erasure and $\tau(q-1)$ errors.

Prop. The Reed–Muller code $\mathrm{RM}_q(r,m)$ is locally correctable, with **locality** $\ell = q-1$, for all errors of relative weight $\delta < \tau/2$, and with soundness δ/τ.

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Let $\boldsymbol{x}$ be a coordinate to be corrected.
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\mathbb{P}(\text{decoding fails at } \boldsymbol{x}) &\leq \mathbb{P}_L(\#\text{errors in } L > \tau(q-1)) \\
&\leq \frac{\mathbb{E}_B[\#\text{errors in } L]}{\tau(q-1)+1} && \text{(Markov's inequality)} \\
&< \frac{\delta(q-1)}{\tau(q-1)} = \frac{\delta}{\tau} && \text{(linearity of } \mathbb{E}\text{)}
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&\leq \frac{\mathbb{E}_B[\#\text{errors in } L]}{\tau(q-1)+1} && \text{(Markov's inequality)} \\
&< \frac{\delta(q-1)}{\tau(q-1)} = \frac{\delta}{\tau} && \text{(linearity of } \mathbb{E}\text{)}
\end{aligned}$$

Parameters:

Local correction of Reed–Muller codes (2)

Let
$$\tau = \frac{q-r-2}{2(q-1)} > 0$$
be such that the Reed-Solomon code $\mathrm{RS}_q(r+1)$ can correct 1 erasure and $\tau(q-1)$ errors.

Prop. The Reed–Muller code $\mathrm{RM}_q(r,m)$ is locally correctable, with **locality** $\ell = q-1$, for all errors of relative weight $\delta < \tau/2$, and with soundness δ/τ.

Let $\boldsymbol{x}$ be a coordinate to be corrected.

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Parameters:

- Locality $\ell = O(n^{1/m}) = o(n)$
- Information rate

$$\frac{k}{n} \simeq \frac{(r/q)^m}{m!} \simeq \frac{(1-2\tau)^m}{m!} < \frac{1}{m!} \cdots$$

Looking for higher rates (1)

LCC with sublinear locality and rate $\frac{k}{n} \geq 1/2$.

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New affine-invariant codes from lifting. A. Guo, S. Kopparty, M. Sudan. ITCS. **2013**.

1. **Affine-invariant "lifted codes":**

$$\mathcal{C} = \left\{ \mathrm{ev}_{\mathbb{F}_q^m}(f) \mid f \in \mathbb{F}_q[X_1, \dots, X_m], \forall \text{ line } L, ev_L(f) \in \mathrm{RS}_q(r) \right\} \subseteq \mathbb{F}_q^{q^m} .$$

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They share the **same local properties** as Reed–Muller codes.
Quite surprisingly, in some cases, they reach much higher rates than RM codes ($k/n \to 1$ in some contexts).

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High-Rate Codes with Sublinear-Time Decoding. S. Kopparty, S. Saraf, S. Yekhanin. J. ACM. **2014**.

2. Multiplicity codes: evaluation of m-variate polynomials **and their Hasse derivative** up to some order s.

$$\mathcal{C} = \left\{ \mathrm{ev}^{<s}_{\mathbb{F}_q^m}(f) \mid f \in \mathbb{F}_q[X_1, \dots, X_m] \right\} \subseteq (\mathbb{F}_q^u)^{q^m}, \quad \text{where } u := \binom{m+s}{s}$$

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Idea: Reed–Muller codes require $r \leq q-1$ for locality; multiplicity codes allow $r \leq s(q-1)$.

Reach higher rates than lifted codes.

Fast encoding algorithm.

Minor drawback: large alphabet.

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Idea: Reed–Muller codes require $r \leq q-1$ for locality; multiplicity codes allow $r \leq s(q-1)$.

- Reach higher rates than lifted codes.
- Fast encoding algorithm.
- Minor drawback: large alphabet.

Rem. Can be mixed up with lifted codes: "lifted multiplicity codes".

Looking for higher rates (3)

LCC with sublinear locality and rate $\frac{k}{n} \geq 1/2$.

LCC with sublinear locality and rate $\frac{k}{n} \geq 1/2$.

Local Correctability of Expander Codes. B. Hemenway, R. Ostrovsky, M. Wootters. Inf. Comp.. **2015**.

3. Some families of **expander codes:** codes indexed by edges E of a d-regular expander graph $\mathcal{G} = (V, E)$, with local constraints in the neighbourhood of each vertex.

$$\mathcal{C}(\mathcal{G}, \mathcal{C}_0) = \left\{ \boldsymbol{c} \in \mathbb{F}_q^E \mid \forall v \in V, \boldsymbol{c}_{|\Gamma(v)} \in \mathcal{C}_0 \right\}$$

where $\mathcal{C}_0$ is a small "local code" and $\Gamma(v)$ is the set of edges adjacent to $v \in V$.

Rem. if $\mathcal{C}_0$ has rate R_0, then $\mathcal{C}(\mathcal{G}, \mathcal{C}_0)$ has rate $2R_0 - 1$.

Definition. A *t-design* of parameters (n, ℓ, λ) consists in:

- a set X of points, $|X| = n$,
- a set $\mathcal{B}$ of blocks $B \subset X$, $|B| = \ell$

such that every t-set in X appears in exactly λ blocks.

Block designs

Definition. A *t-design* of parameters (n, ℓ, λ) consists in:

- a set X of points, $|X| = n$,
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such that every t-set in X appears in exactly λ blocks.

Incidence matrix of a design:

$$\text{blocks in } \mathcal{B} \left\{ \overbrace{\begin{pmatrix} & & & & * & & & & \\ 0 & \cdots & 0 & \mathbf{1} & \cdots & \mathbf{1} & 0 & \cdots & 0 \\ & & & & * & & & & \end{pmatrix}}^{\text{points in } X} \right. \longleftarrow \text{indicator vector of } \textbf{block } B$$

Def. The **code based on the design** $\mathcal{D} = (X, \mathcal{B})$ is the code $\mathcal{C} = \mathrm{Code}(\mathcal{D}) \subseteq \mathbb{F}_q^X$ admitting the incidence matrix of $\mathcal{D}$ as a parity-check matrix. In other words:

$$\mathrm{Code}(\mathcal{D}) = \{\boldsymbol{c} \in \mathbb{F}_q^X \mid \forall B \in \mathcal{B}, \boldsymbol{c}_{|B} \in \mathrm{Parity}\}$$

Remark. The dimension of $\mathrm{Code}(\mathcal{D})$ is highly dependent on the field $\mathbb{F}_q$

Codes based on designs, and generalization

Def. The **code based on the design** $\mathcal{D} = (X, \mathcal{B})$ is the code $\mathcal{C} = \mathrm{Code}(\mathcal{D}) \subseteq \mathbb{F}_q^X$ admitting the incidence matrix of $\mathcal{D}$ as a parity-check matrix. In other words:

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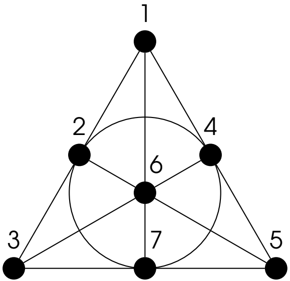
Remark. The dimension of $\mathrm{Code}(\mathcal{D})$ is highly dependent on the field $\mathbb{F}_q$

Def. Let $(\mathcal{C}_B)_{B \in \mathcal{B}}$ be a family of codes indexed by blocks $B \in \mathcal{B}$. The **generalised design-based code** based on $(\mathcal{D}, \mathcal{B})$ is

$$\mathrm{Code}(\mathcal{D}, (\mathcal{C}_B)_B) := \{\boldsymbol{c} \in \mathbb{F}_q^X \mid \forall B \in \mathcal{B}, \boldsymbol{c}_{|B} \in \mathcal{C}_B\}.$$

Design-based codes: an example

Fano plane $\mathcal{D}$



Corresponding incidence matrix

$$\boldsymbol{H} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

The design-based code $\mathcal{C} = \mathrm{Code}(\mathcal{D})$ is:

- a $[7, 3, 4]_2$ code if $q = 2$ (simplex code);
- a $[7, 1, 7]_2$ code if $q = 3$;
- the zero code for other primes q.

Design-based codes are locally correctable

Generalised design-based code $\mathcal{C} = \mathrm{Code}(\mathcal{D}, (\mathcal{C}_B)_B)$, where

- $\mathcal{D} = (X, \mathcal{B})$ is a t-$(n, \ell + 1, \lambda)$-design
- $\tau \in (0, \frac{1}{2})$ is fixed
- $(\mathcal{C}_B)_{B \in \mathcal{B}}$ is such that each code in $\mathcal{C}_B$ corrects a fraction τ of errors.

Algorithm. Local correction of $\boldsymbol{y} \in \mathbb{F}_q^X$ at $x \in X$

1. Pick uniformly at random a block $B \in \mathcal{B}$ such that $x \in B$.
2. Correct $\boldsymbol{y}_{|B}$ as a noisy codeword from $\mathcal{C}_B$, and output the (corrected) symbol y_x.

Design-based codes are locally correctable

Proposition ($t = 2$). For every $\delta < \tau/2$, $\mathrm{Code}(\mathcal{D}, (\mathcal{C}_B)_B)$ is a $(\ell, \delta, \varepsilon)$-LCC, where

$$\varepsilon = \delta/\tau\,.$$

Let i be a coordinate to be corrected.

$$\begin{aligned}
\mathbb{P}(\text{decoding fails at } i) &\le \mathbb{P}_B(\#\text{errors in } B_i \ge \tau\ell) \\
&\le \frac{\mathbb{E}_B[\#\text{errors in } B_i]}{\tau\ell} && \text{(Markov's inequality)} \\
&\lesssim \frac{\delta\ell}{\tau\ell} && \text{(linearity of } \mathbb{E}\text{)}
\end{aligned}$$

Proposition ($t = 3$). For every $\delta < \tau - 1/\sqrt{2\ell}$, $\mathrm{Code}(\mathcal{D}, (\mathcal{C}_B)_B)$ is a $(\ell, \delta, \varepsilon)$-LCC where

$$\varepsilon = \frac{\delta(1-\delta)}{(\tau-\delta)^2} \cdot \frac{1}{\ell} \le \frac{1}{\tau^2\ell} \cdot \delta\,.$$

(proof: Bienaymé–Tchebychev inequality)

Perspectives

- Designs-based codes with large dimension?
- Applications in crypto, *e.g.* private information retrieval