Private information retrieval with a coding-theoretic perspective

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Séminaire de l'équipe ECO

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1. Private information retrieval

2. PIR schemes with low computation and storage
   - Transversal designs and codes
   - A PIR scheme with transversal designs
   - Collusion-resistant PIR schemes with weighted lifted codes

3. PIR schemes for common storage systems
   - Distributed storage systems
   - A PIR scheme on RS-coded databases
   - A PIR scheme with regenerating codes

Problem statement

Private information retrieval (PIR):

Given a **remote** database $F \in \Sigma^M$ and $i \in [1, M]$,
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Trivial solution: full download.

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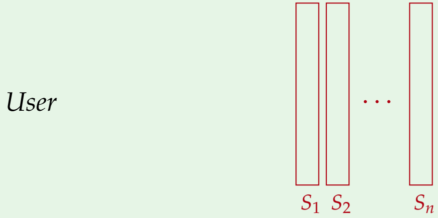

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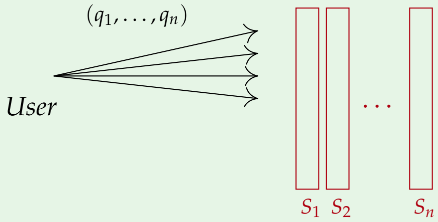

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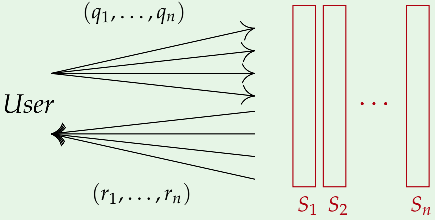

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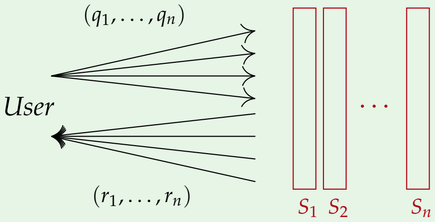
3. U recovers the desired entry
   $$F_i = \mathcal{R}(\boldsymbol{q}, \boldsymbol{r}, i)\,.$$



Privacy

The adversary: a **collusion of servers** = a subset of servers $\{S_j : j \in T\}$, where $T \subset [1, n]$, which exchange information about queries, data, etc.

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- **Information-theoretic (IT) privacy:**

$$\mathrm{I}(i\,; \boldsymbol{q}_{|T}) = 0, \quad \forall T \subseteq [1, n], |T| \leq t.$$

- **Computational privacy:** by varying the index i, distributions of queries $\boldsymbol{q}_{|T} = \mathcal{Q}(i)_{|T}$ are computationally indistinguishable.

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Theorem [CGKS95, CG97]. If $t = n$ (in particular if $n = 1$ server), then:

- for **IT** privacy, **no better solution than full download**,
- **computational** privacy is possible, but remains **expensive** as of now.

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- **replicated** database vs. **coded** database
- unresponsive or **byzantine** servers
- small entries vs. large entries
- bounded vs. unbounded number of entries in the database
- dynamic database vs. static database

Private Information Retrieval. Chor, Goldreich, Kushilevitz, Sudan. FOCS. **1995**.

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- $|F| = M$ entries (bits), with $M = L^2$, and $[1, M] \simeq [1, L]^2$.
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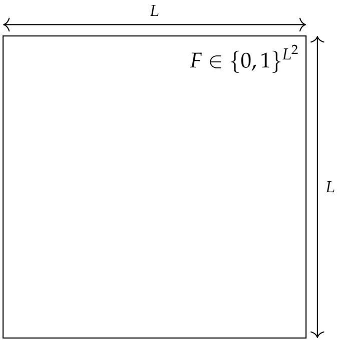

Seminal work [CGKS'95-98]

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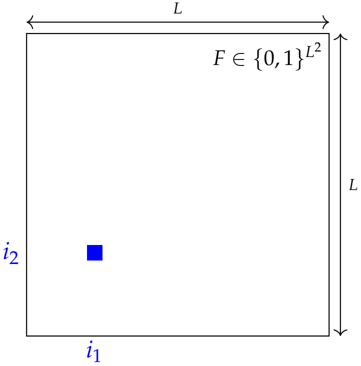

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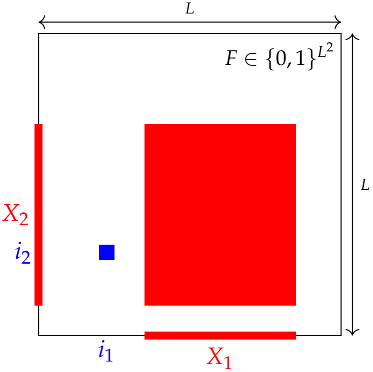

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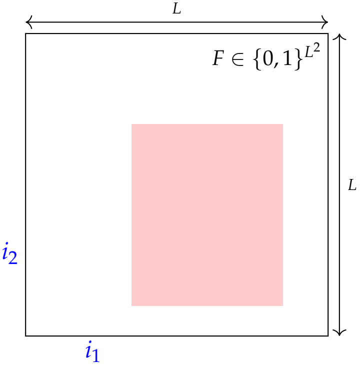

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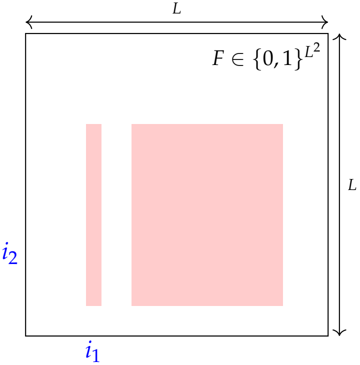

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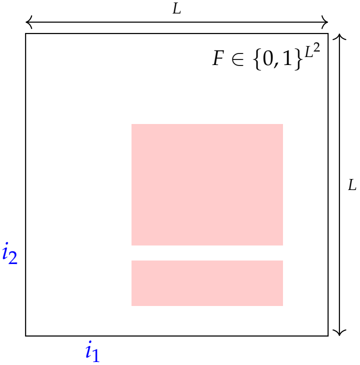

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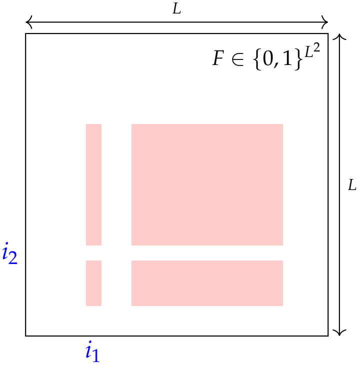

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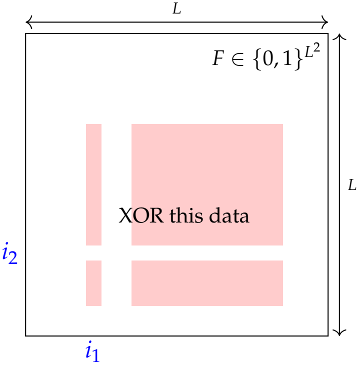

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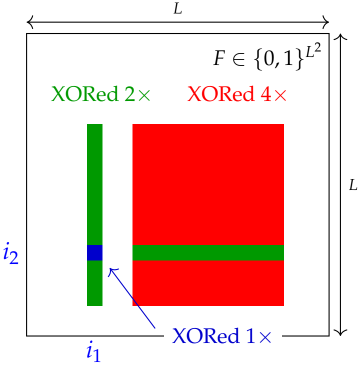

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3. User XORs the 4 bits and retrieves F_i.

Correct, and **secure** if no collusion.

Features of the PIR scheme in [CGKS'95-98]

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With $n = 4$ servers:

- **Communication:** $8\sqrt{M}$ uploaded bits, 4 downloaded bits,
- **Storage:** replication of F over $n = 4$ servers,
- **Complexity:**
  - for each server: in average, XOR of $(L/2)^2 = M/4$ bits
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Generalisable to $n = 2^b$ servers:

- **Communication:** $b2^b M^{1/b} = n\log(n) M^{1/\log(n)}$ uploaded bits, n downloaded bits,
- **Storage:** replication of F over n servers,
- **Complexity:**
  - for each server: in average, XOR of M/n bits
  - for the user: XOR of n bits.

(Short) state of the art

- 1995: first definition [CGKS95]
- 2000: reduction from smooth locally decodable codes [KT00]
- 2000-10's: many improvements
  - PIR with 3 servers and subpolynomial communication [Yek08, Efr09]
  - PIR with 2 servers and subpolynomial communication [DG16]
  - lower storage overhead with *PIR codes* [FVY15]
- 2016-now: capacity-achieving schemes, schemes dedicated to storage systems
  - capacity of PIR [SJ17, BU18]
  - (nearly) capacity-achieving schemes [SRR14, CHY15, TR16, ...]

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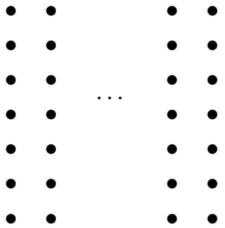
Tools: coding theory

- codes from transversal designs
- “lifted” codes

A **transversal design** $\mathrm{TD}(n, s) = (X, \mathcal{B}, \mathcal{G})$ is given by:

- X a set of *points*, $|X| = N = ns$,



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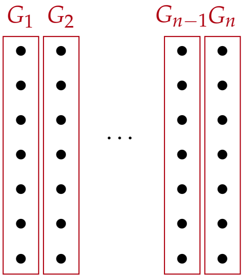

Transversal designs

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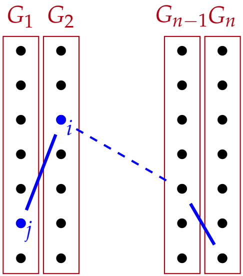
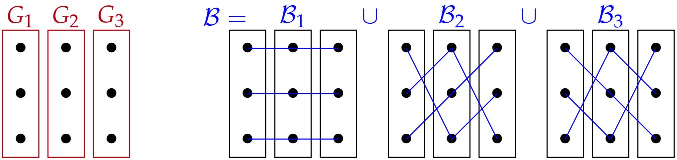
- *blocks* $B \in \mathcal{B}$ satisfying
  - $B \subset X$ and $|B| = n$;
  - for all $\{i,j\} \subset X$, $\{i,j\}$ lie:
    either in a single group $G \in \mathcal{G}$,
    or in a unique block $B \in \mathcal{B}$



An example for a $\mathrm{TD}(3,3)$:

- $ns = 9$ points
- $s = 3$ groups G_1, G_2, G_3 of size 3
- $ns = 9$ blocks of $n = 3$ points, partitionned into 3 parallel classes $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$



Codes from designs

Let $\mathcal{T}$ be a transversal design $\mathrm{TD}(n, s) = (X, \mathcal{B}, \mathcal{G})$.

Its **incidence matrix** $\boldsymbol{M}$ has size $|\mathcal{B}| \times |X| = ns \times ns$, and is defined by:

$$M_{i,j} = \begin{cases} 1 & \text{if } x_j \in B_i \\ 0 & \text{otherwise.} \end{cases}$$

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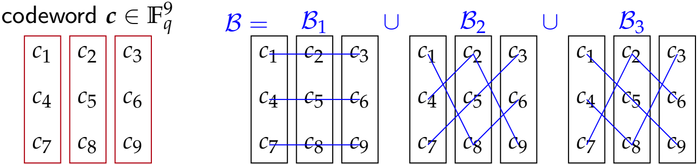
Definition. The **code $\mathcal{C}$ based on $\mathcal{T}$ over $\mathbb{F}_q$** is the $\mathbb{F}_q$-linear code having $\boldsymbol{M}$ as a parity-check matrix (*i.e.* $\mathcal{C}^\perp$ is generated by $\boldsymbol{M}$).

- $\mathrm{length}(\mathcal{C}) = |X| = ns$,
- $\dim(\mathcal{C}) = \dim(\ker \boldsymbol{M})$,
- every block $B \in \mathcal{B}$ gives a parity-check equation $\boldsymbol{h} \in \mathcal{C}^\perp$, such that

$$\mathrm{wt}(\boldsymbol{h}_{|G_j}) = 1, \quad \forall j = 1, \dots, n$$

Example

The transversal design TD$(3,3)$ represented by:



gives a code with the following parity-check matrix:

$$
\boldsymbol{H} = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
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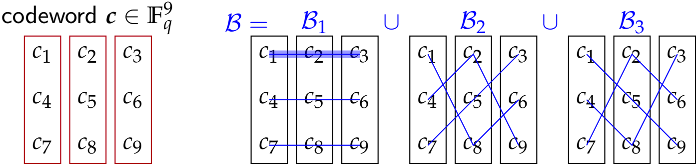
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

Example

The transversal design $\mathrm{TD}(3,3)$ represented by:



gives a code with the following parity-check matrix:

$$\boldsymbol{H} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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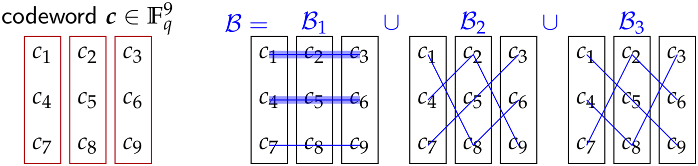

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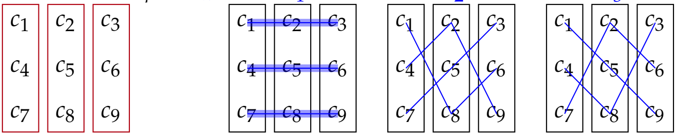
The transversal design $\mathrm{TD}(3,3)$ represented by:

codeword $\boldsymbol{c} \in \mathbb{F}_q^9$ $\quad\quad$ $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$



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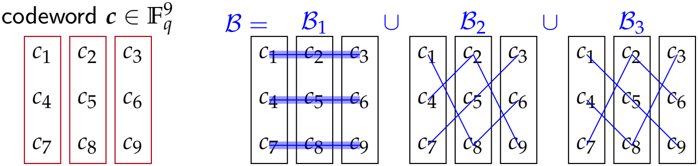

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Dimension of the code?

- depends on q
- for instance, over $\mathbb{F}_3$, we have $\mathrm{rk}(\boldsymbol{H}) = 6 \quad \dim(\mathcal{C}) = 3$.

Outline

The PIR scheme

Private Information Retrieval from Transversal Designs. **L.**. IEEE-TIT. **2019**.

Let $\mathcal{C} \subseteq \mathbb{F}_q^N$ be a code based on a $\mathrm{TD}(n, s)$, with $N = ns$.

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  1. User U randomly picks a block $B \in \mathcal{B}$ containing i.
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$$q_j = \mathcal{Q}(i)_j := \begin{cases} \text{unique } \in B \cap G_j & \text{if } i \notin G_j \\ \text{a random point in } G_j & \text{otherwise.} \end{cases}$$

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  2. Each server S_j sends back c_{q_j}
  3. U recovers

$$c_i = -\sum_{j:\, i \notin G_j} c_{q_j} = -\sum_{b \in B \setminus \{i\}} c_b$$

Privacy and parameters

Theorem. This PIR protocol is information-theoretically private.

Proof:

- the only server which holds F_i received a random query;
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Question: transversal designs leading to large dimension codes?

An example: the **classical affine transversal design**:

- $X = \mathbb{F}_q^m$ for $m \geq 2$,
- $\mathcal{G}$ a partition of X into q hyperplanes $G_1, \dots, G_q$,
- $\mathcal{B} = \{\text{affine lines } L \text{ secant to each } G_j\}$.

The code has:

- length $ns = q^m$,
- "locality" $n = q$.

Instances with geometric designs

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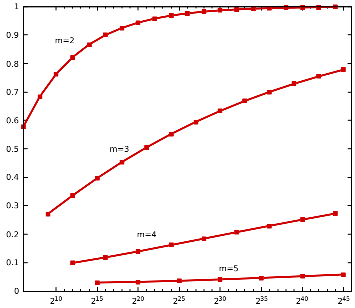

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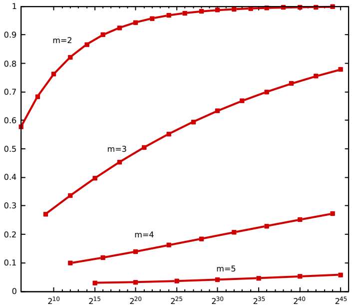

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Definition. The (full-length) **Reed–Solomon code** of dimension k over $\mathbb{F}_q$ is:

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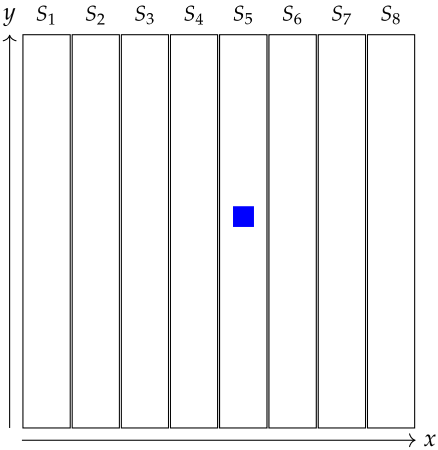
Reed–Muller codes have the following property:

$$\forall \boldsymbol{c} = \mathrm{ev}_{\mathbb{A}^m}(f) \in \mathrm{RM}_q(m,r) \quad \text{and} \quad \forall \text{ affine line } L \subset \mathbb{A}^m,$$
$$\mathrm{ev}_{\mathbb{A}^1}(f_{|L}) \in \mathrm{RS}_q(r+1)\,.$$

(where $f_{|L}$ is the lowest-degree univariate polynomial interpolating f over L)

A PIR scheme based on Reed–Muller codes



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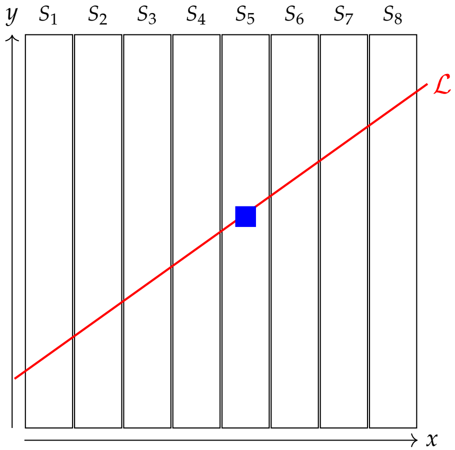

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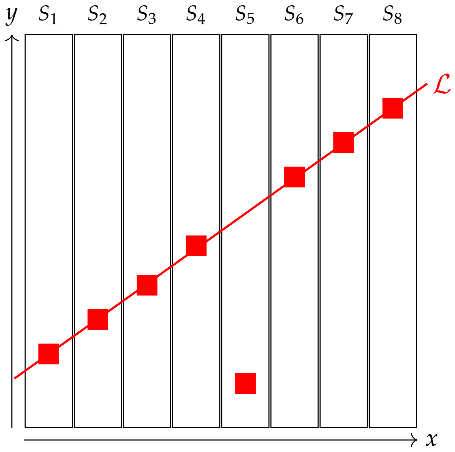

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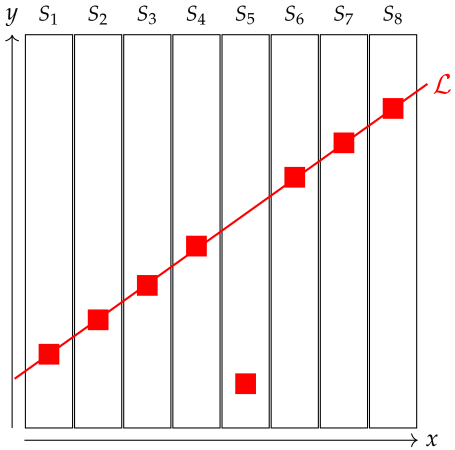

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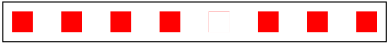

received answer

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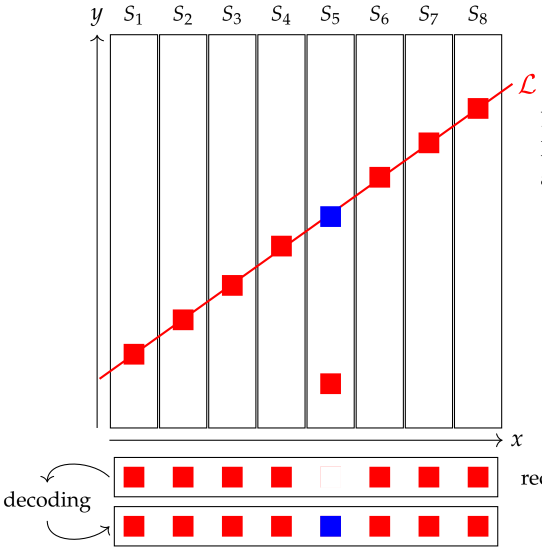

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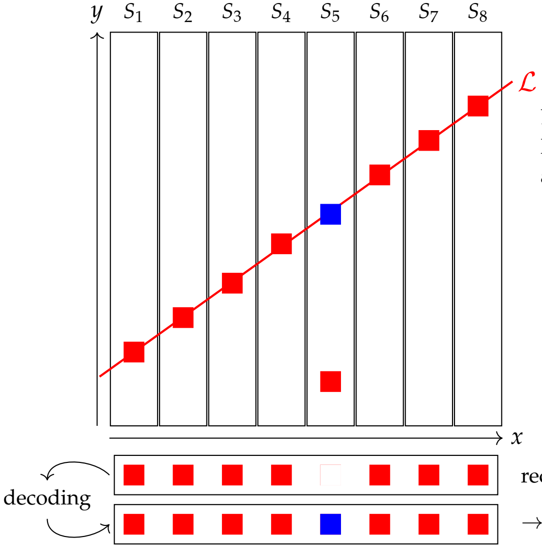

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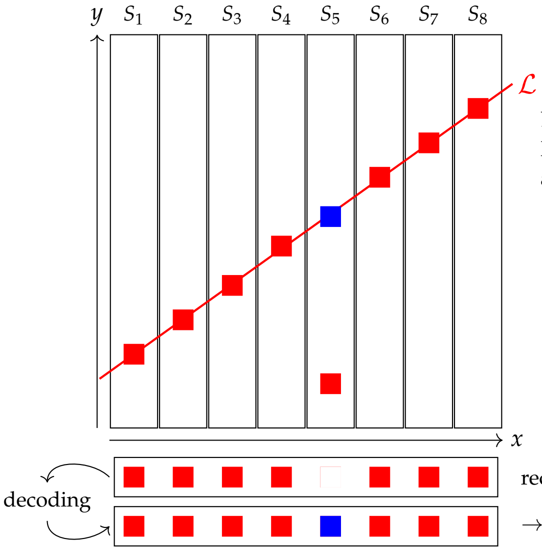

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$\mathcal{L}$ is an affine line

$\Longrightarrow$ no individual server can find where is ■

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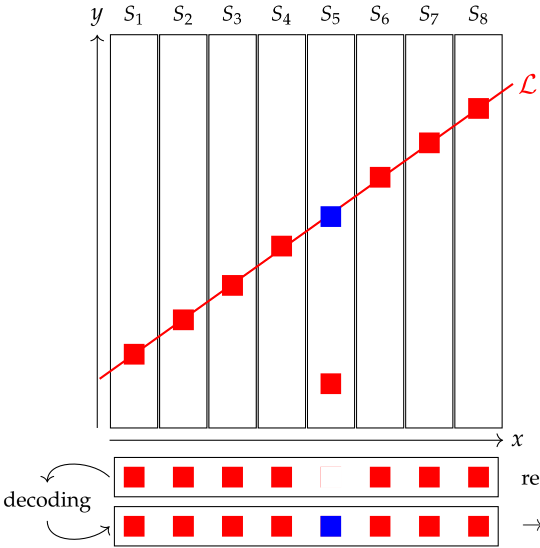

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$\implies$ the PIR scheme is private

Parameter of the Reed–Muller code-based PIR

Features with $\mathrm{RM}_q(m, r)$ of length q^m.

- communication complexity: $(m-1)q \log q$ uploaded bits, $q \log q$ downloaded bits
- computational complexity:
  - **only 1 read for each server** (optimal)
  - a decoding procedure for $\mathrm{RS}_q(r)$ for the user
- storage overhead: the rate of $\mathrm{RM}_q(m, r)$ with $r \leq q-1$ is

$$\simeq \frac{(r/q)^m}{m!} \ldots$$

$\Longrightarrow$ We need codes with the same properties, but larger dimension.

Lifted codes

"Lifted" codes are the largest codes having the same property as Reed–Muller codes.

New affine-invariant codes from lifting. Guo, Kopparty, Sudan. ITCS. **2013**.

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Definition. The ***m*-th lifted Reed-Solomon code** of degree r over $\mathbb{F}_q$ is:

$$\mathrm{Lift}_q(m,r) := \{\mathrm{ev}_{\mathbb{A}^m}(f) \mid f \in \mathbb{F}_q[\mathbf{X}] \text{ and } \forall \text{ affine line } L \subset \mathbb{A}^m, \deg(f_{|L}) \leq r\}.$$

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Example. For $q = 4$, $m = 2$, $r = 2$, consider $f(X,Y) = X^2Y^2$.

$$f(aT+b, cT+d) \;\equiv\; (a^2d^2 + b^2c^2)T^2 + a^2c^2T + b^2d^2 \mod (T^4 - T)$$

Hence,

$$\mathrm{ev}(X^2Y^2) \in \mathrm{Lift}_4(2,2) \quad \text{but} \quad \mathrm{ev}(X^2Y^2) \notin \mathrm{RM}_4(2,2)\,.$$

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Fact. For every m, lifted codes reach arbitrarily large information rates.

Weighted lifted codes

Question: how to deal with collusions and byzantine errors?

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For convenience, here $m = 2$.

Definition. A t-curve is:

$$\mathcal{L} = \{(x, g(x)) \in \mathbb{A}^2 \mid g \in \mathbb{F}_q[X], \deg(g) \leq t\}$$

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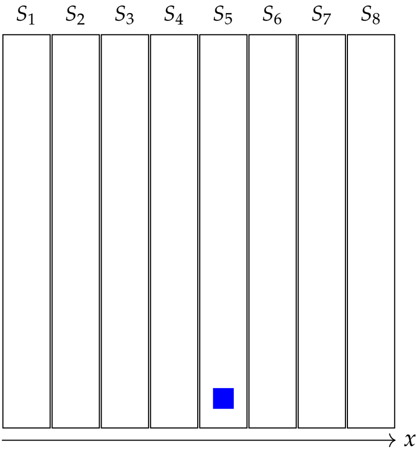
Definition. The **weighted lifted Reed-Solomon code** of degree r and weight t over $\mathbb{F}_q$ is:

$$\mathrm{WLift}_q(t, r) := \{\mathrm{ev}_{\mathbb{A}^2}(f) \mid f \in \mathbb{F}_q[X, Y] \text{ and } \forall\ t\text{-curve } \mathcal{L} \subset \mathbb{A}^2, \deg(f_{|\mathcal{L}}) \leq r\}$$

Consequence: for every codeword $\boldsymbol{c} \in \mathrm{WLift}_q(t, r)$ and every t-curve $\mathcal{L}$, we have:

$$\boldsymbol{c}_{|\mathcal{L}} \in \mathrm{RS}_q(r + 1)\,.$$

A PIR scheme based on weighted lifted codes



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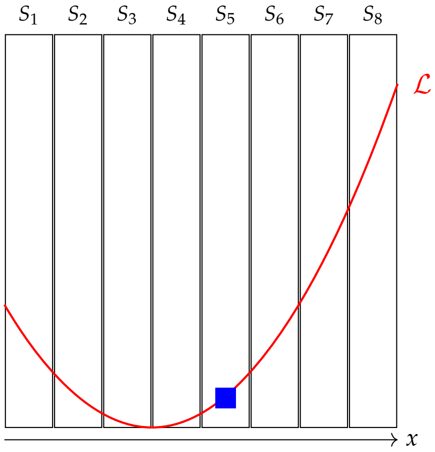

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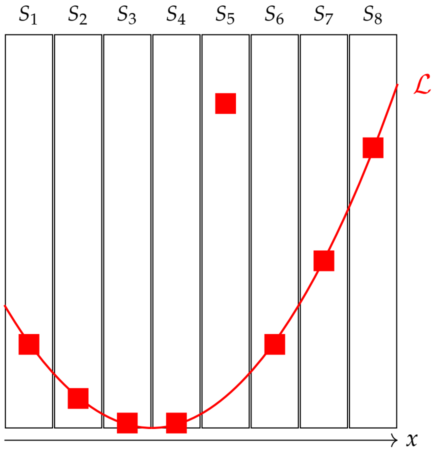

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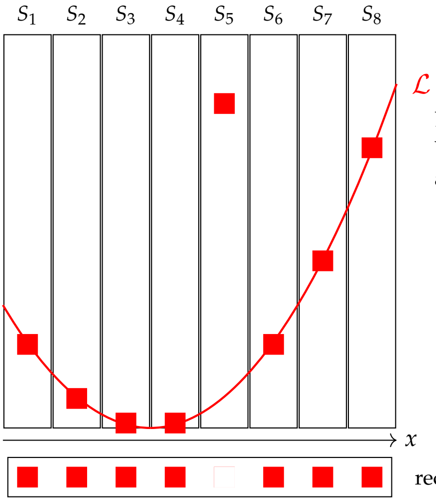

Database F in encoded with $\mathrm{WLift}_q(t, r)$, then distributed across the servers

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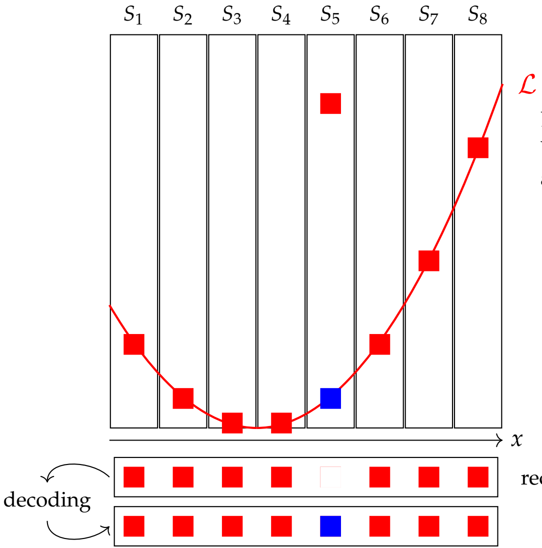

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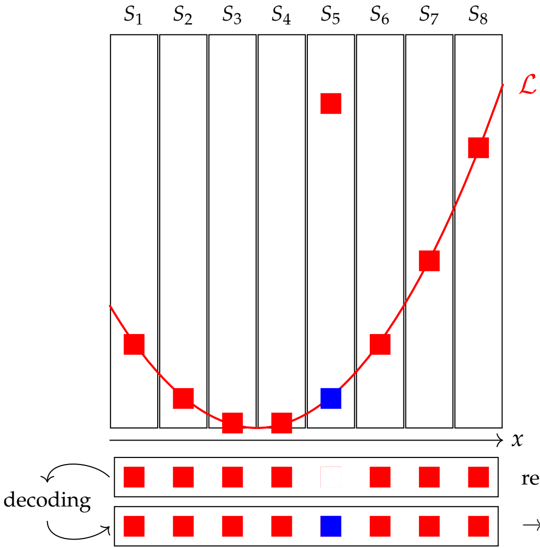

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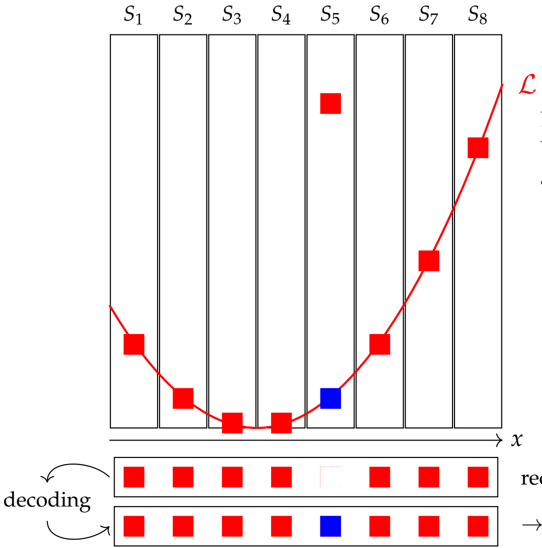

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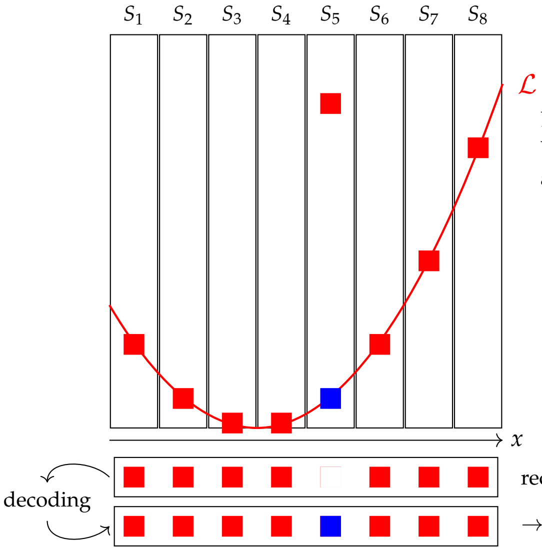

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$\Longrightarrow$ no t-set of servers can find where is ■

$\Longrightarrow$ the PIR scheme is t-private

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Parameters

Weighted Lifted Codes: Local Correctabilities and Application to Robust Private Information Retrieval. **L.**, Nardi. IEEE TIT. **2021**.

Theorem. Let p be a prime number, $t \geq 1$ and $\alpha \geq 2$ be fixed integers. Then, the information rate $\mathrm{WLift}_{p^e}(t, p^e - \alpha)$ grows to 1 when $e \to \infty$.

Corollary: we get PIR schemes with relative storage overhead $\to 0$, for a constant number of adversaries.

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Theorem. Let p be a prime number, $t \geq 1$ and $c \geq 1$ be fixed integers. Let $\gamma = 1 - p^{-c}$ and $\mathcal{C}_e = \mathrm{WLift}_{p^e}(t, \gamma p^e)$. Then, the information rate R_e of $\mathcal{C}_e$ satisfies:

$$\lim_{e \to \infty} R_e = K_{t,p,c} > 0$$

Corollary: we get PIR schemes with **constant relative storage overhead**, for a **constant number of collusions** and a **constant fraction of errors**.

Outline

Context

Storage systems use codes to cope with node failures.

- Before 2010: mostly replication or parity-check.
- 2010's: MDS storage (*e.g.* $[14, 10]$ Reed-Solomon code for Facebook).
- Recently: codes with locality (*e.g.* Hadoop Xorbas).

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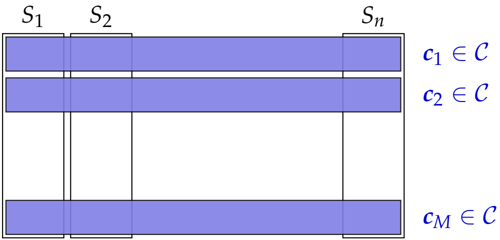
Given a code $\mathcal{C}$ of length n:



Definition (Reed-Solomon code). Let $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{F}_q^n$, pairwise distinct.

$$\mathrm{RS}_q(k,n) := \{(f(x_1), \dots, f(x_n)), f \in \mathbb{F}_q[X], \deg f < k\}$$

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File storage:

$$\text{a file} \quad F_i \in \Sigma \simeq \mathbb{F}_{q^s}^k \quad \text{is encoded into} \quad \boldsymbol{c}_i \in \mathrm{RS}_q(k,n) \otimes \mathbb{F}_{q^s}$$

Example: Reed-Solomon storage systems

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Main assumption (can be discussed):

$$s \gg M$$

Outline

Usual goal (assuming $s \gg M$): a large *PIR rate*

$$\rho := \frac{|F_i|}{|\boldsymbol{r}|} \, .$$

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Next, we present a PIR scheme for RS-coded databases.

- Originally [TR16], then extended and reformulated [TGKFH18, TGR18].
- Optimal PIR rate for $t = 1$ and $M \to \infty$.
- PIR rate conjectured optimal for $M \to \infty$.

[TR16] *PIR from MDS Coded Data in Distributed Storage Systems*. Tajeddine, El Rouayheb. ISIT. **2016**.

[TGKFH18] *Robust PIR from Coded Systems with Byzantine and Colluding Servers*. Tajeddine, Gnilke, Karpuk, Freij-Hollanti, Hollanti. ISIT. **2018**.

[TGR18] *PIR from MDS Coded Data in Distributed Storage Systems*. Tajeddine, Gnilke, El Rouayheb. IEEE-TIT. **2018**.

Notation:

$$\begin{aligned} \boldsymbol{a} \star \boldsymbol{b} &:= (a_1 b_1, \dots, a_n b_n) \\ \mathcal{C} \star \mathcal{C}' &:= \langle \{ \boldsymbol{c} \star \boldsymbol{c}' \mid \boldsymbol{c} \in \mathcal{C}, \boldsymbol{c}' \in \mathcal{C}' \} \rangle \end{aligned}$$

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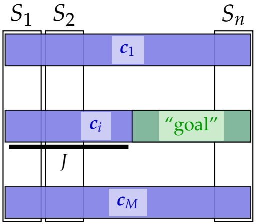

System parameters:

$\mathcal{C} \subseteq \mathbb{F}_q^n$ the *storage code*, $\boldsymbol{C} \in \mathcal{C}^M$ the coded database

$J \subseteq [1, n]$ an information set for $\mathcal{C} \star \mathcal{D}$, and $\bar{J} := [1, n] \setminus J$

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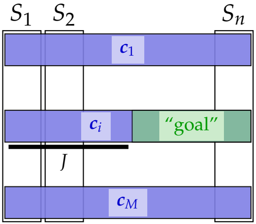

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The protocol: query generation

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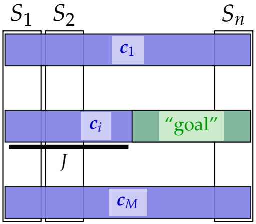

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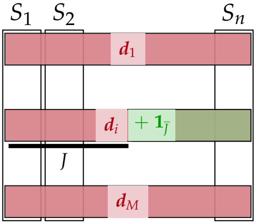
1. the user generates at random M words $\boldsymbol{d}_1, \dots, \boldsymbol{d}_M \in \mathcal{D}$ and defines $\boldsymbol{Q}$ as follows:
2. the j-th column of $\boldsymbol{Q}$ is sent to server S_j



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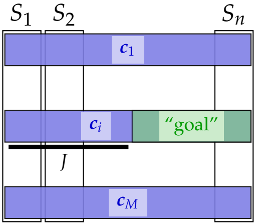
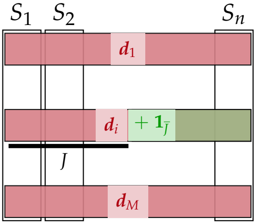
Remark: queries remain private against collusions of servers of size $\leq t$.





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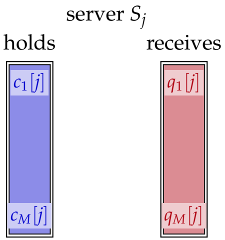

The protocol: server answers and reconstruction

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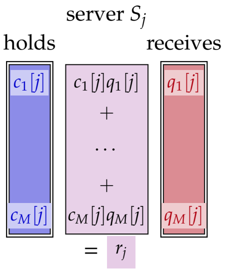

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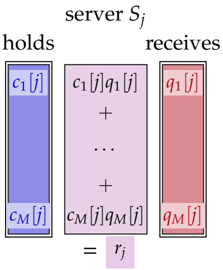

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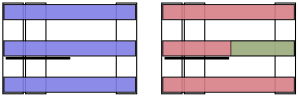

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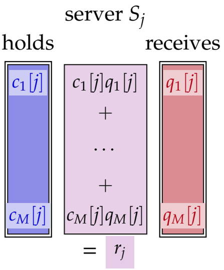
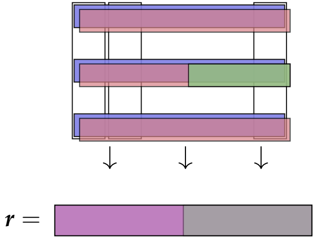


Reconstruction: The user collects

$$\boldsymbol{r} = (r_1, \dots, r_n) = \underbrace{\sum_{m=1}^{M} \boldsymbol{d}_m \star \boldsymbol{c}_m}_{\in \mathcal{C} \star \mathcal{D}} + \underbrace{\boldsymbol{1}_{\bar{J}} \star \boldsymbol{c}_i}_{= \boldsymbol{c}_i \text{ on } \bar{J}}$$



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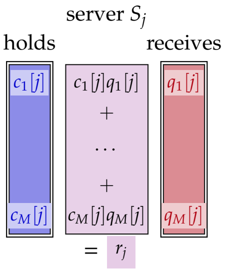

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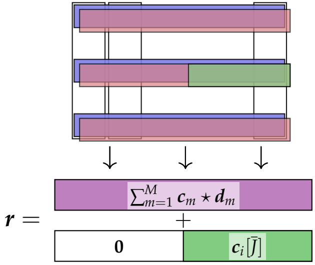
and interpolates on J to recover

- $\sum_{m=1}^{M} \boldsymbol{d}_m \star \boldsymbol{c}_m$,
- then $\boldsymbol{c}_i[|\bar{J}]$.



Features for 1 run of the protocol.

- download cost: n symbols over $\mathbb{F}_{q^s}$
- upload cost: an $(M \times n)$-matrix over $\mathbb{F}_q$ (negligible if $s \gg M$)
- retrieval of $|\bar{J}| = n - \dim(\mathcal{C} \star \mathcal{D})$ symbols of the desired file
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Definition: $\mathcal{C}$ is an $(n, k, d, \alpha, \beta, B)$-**regenerating code** if:

- $\mathcal{C}$ is a linear space of dimension B, consisting in $(\alpha \times n)$-matrices over $\mathbb{F}_q$,
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A particular optimal point (minimum-bandwidth repair, MBR): $d\beta = \alpha$.
Then,

$$B = \left(kd - \frac{k(k-1)}{2}\right)\beta\,.$$

Product-matrix MBR codes [RSK11]

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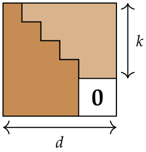
1. Message symbols are arranged in a $(d \times d)$-matrix

$$A = \begin{pmatrix} \boldsymbol{S} & \boldsymbol{T}^\top \\ \boldsymbol{T} & \boldsymbol{0} \end{pmatrix}$$

   where $\boldsymbol{S}$ is $(k \times k)$-symmetric.



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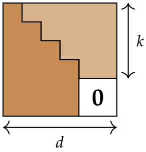

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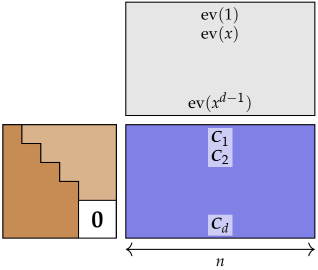

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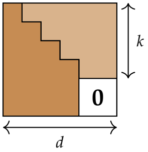

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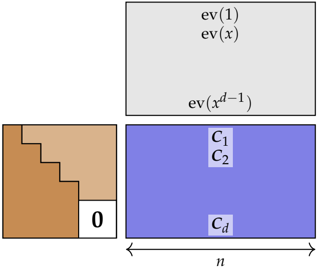
$$C = AG \in \mathbb{F}_q^{d \times n} .$$



Remark: row C_j of C is a word of a RS code

- of dimension k, if $j > k$,
- of dimension $d > k$ otherwise.

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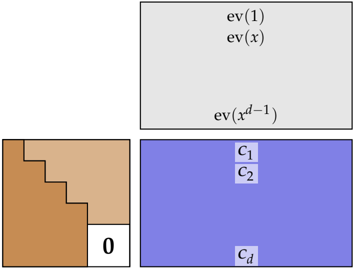
PIR scheme with **no collusion** ($t = 1$).



PIR scheme on PM-MBR codes with no collusion

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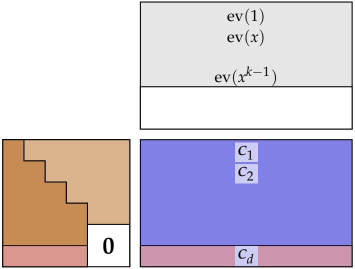

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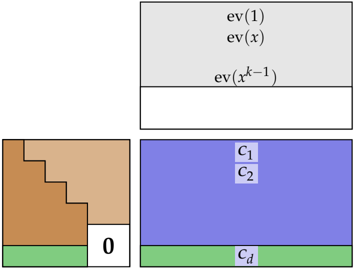

Retrieval rate: $1 - \frac{k}{n}$

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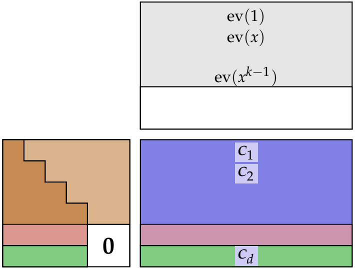

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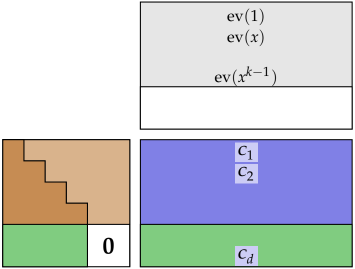

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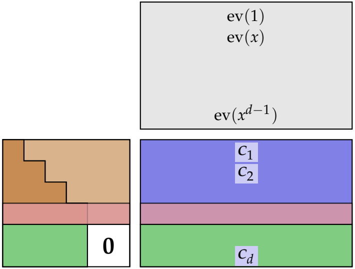

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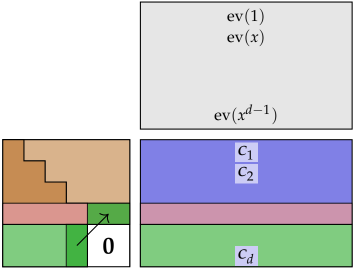

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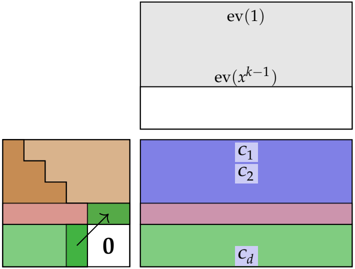

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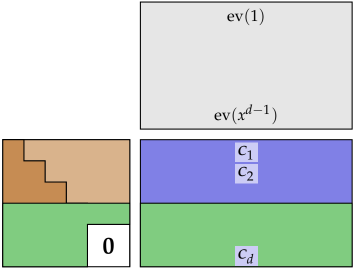

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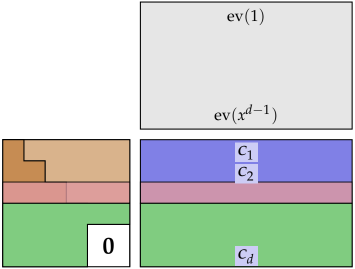

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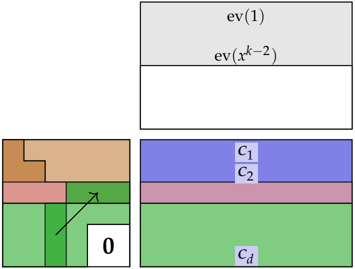

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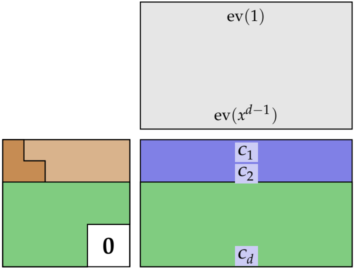

Retrieval rate: $1 - \frac{k-1}{n}$

PIR scheme on PM-MBR codes with no collusion

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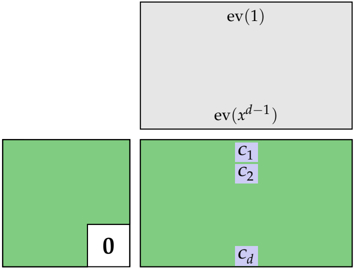

Retrieval rate: $1 - \frac{j}{n}$

PIR scheme on PM-MBR codes

We get a PIR rate:

$$\rho = \frac{1-\frac{k}{n}}{1-\frac{k(k+1)(k-1)}{nB}} > 1-\frac{k}{n}$$

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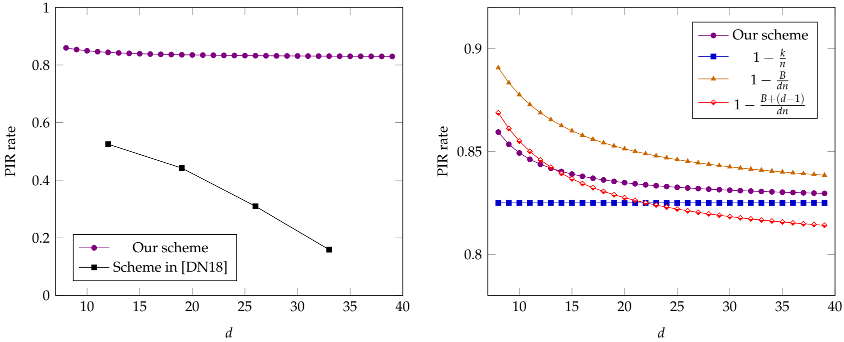

Comparison of PIR rates for $n = 40$ and $k = 7$.