On the construction of PIR schemes

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Outline

1. Private information retrieval

2. PIR schemes for common storage systems
   - Distributed storage systems
   - A PIR scheme on RS-coded databases
   - A PIR scheme with regenerating codes

3. PIR schemes with low computation
   - Transversal designs and codes
   - A PIR scheme with transversal designs
   - Instances

4. Conclusion

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Trivial solution: full download.

Definition of PIR

Introduced in:

Private Information Retrieval. Chor, Goldreich, Kushilevitz, Sudan. FOCS. **1995**.

Database F stored (in some way) on n servers $S_1, \dots, S_n$,
user U wants to recover F_i privately.

A **Private Information Retrieval protocol** is a set of algorithms $(\mathcal{Q}, \mathcal{A}, \mathcal{R})$:

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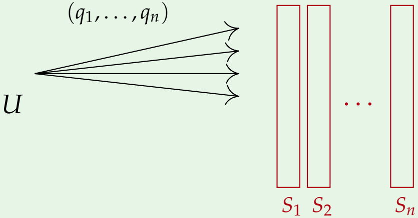

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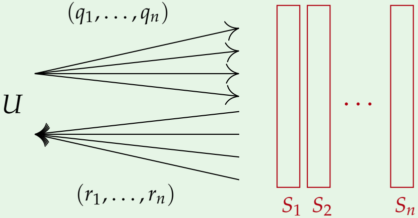

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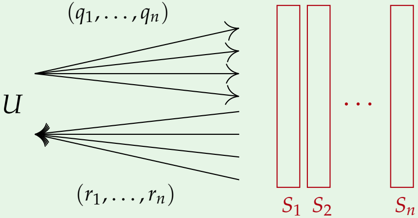
3. U recovers $F_i = \mathcal{R}(\boldsymbol{q}, \boldsymbol{r}, i)$



Privacy

A **collusion of servers**: set of servers $\{S_j : j \in T\}$, where $T \subset [1, n]$, which exchange information about queries, data, etc.

$$t := \max\{|T|, T \subseteq [1, n] \text{ is a collusion}\} \geq 1$$

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- **Information-theoretic privacy:**

$$\mathrm{I}(i; \boldsymbol{q}_{|T}) = 0, \quad \forall T \subseteq [1, n], |T| \leq t.$$

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Theorem [CGKS95, CG97]. If $t = n$ (in particular if $n = 1$), then:

- for IT-privacy, **no better solution than full download**,
- computational privacy is possible (but remains **expensive** as of now).

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(hence we need $n \geq 2$ servers)

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Parameters to be taken into account:

- **communication complexity** (upload and download)
- computation complexity (client and servers)
- global server storage overhead
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Several possible **settings**:

- bounded vs. unbounded number of entries in the database
- replicated database vs. coded database
- small entries vs. large entries
- dynamic database vs. static database
- unresponsive or byzantine servers

Seminal work [CGKS'95-98]

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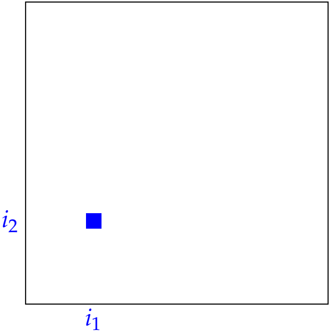
Settings:

- $|F| = M$ bits, with $M = L^2$, and $[1, M] \simeq [1, L]^2$.
- $n = 4$ servers $S_{00}, S_{01}, S_{10}, S_{11}$, each storing a replica of F.
- **Goal:** retrieve $F_i = F_{(i_1, i_2)}$, for $1 \leq i_1, i_2 \leq L$.



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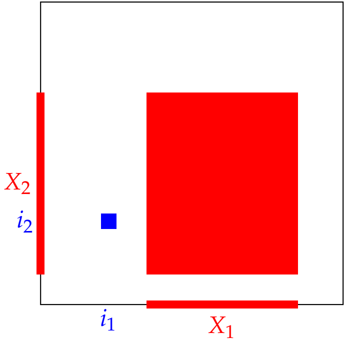

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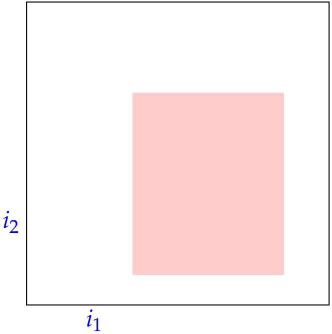

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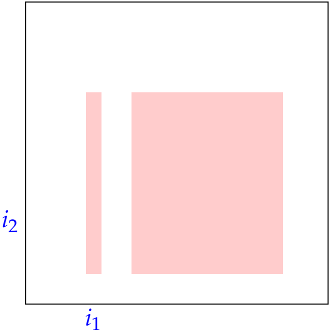

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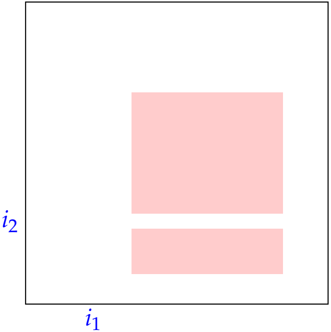

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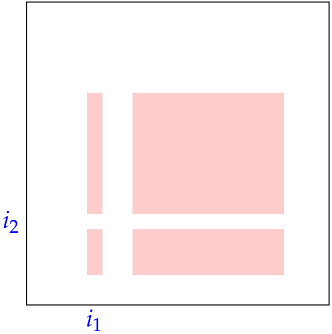

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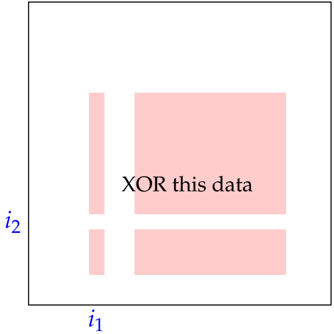

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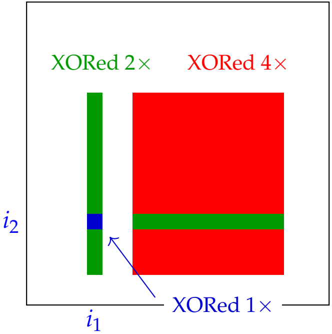

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3. User XORs the 4 bits and retrieves F_i.

Features of the PIR scheme in [CGKS'95-98]

Correct, and **secure** if no collusion.

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With $n = 4$ servers:

- **Communication:** $8\sqrt{M}$ uploaded bits, 4 downloaded bits,
- **Storage:** replication of F over 4 servers,
- **Complexity:**
  - for each server: in average, XOR of $(L/2)^2 = M/4$ bits
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Generalisable to $n = 2^b$ servers:

- **Communication:** $b2^b M^{1/b} = n\log(n)M^{1/\log(n)}$ uploaded bits, n downloaded bits,
- **Storage:** replication of F over n servers,
- **Complexity:**
  - for each server: in average, XOR of M/n bits
  - for the user: XOR of n bits.

(Short) state of the art

- 1995: first definition [CGKS95]

- 2000: reduction from smooth locally decodable codes [KT00]

- 2000-10's: many improvements
  - PIR with 3 servers and subpolynomial communication [Yek08, Efr09]
  - PIR with 2 servers and subpolynomial communication [DG16]
  - lower storage overhead with *PIR codes* [FVY15]

- 2016-now: capacity-achieving schemes, schemes dedicated to storage systems
  - capacity of PIR [SJ17, BU18]
  - (nearly) capacity-achieving schemes [SRR14, CHY15, TR16, ...]

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Context

Storage systems use codes to cope with node failures.

- Before 2010: mostly replication or parity-check.
- 2010's: MDS storage (*e.g.* $[14, 10]$ Reed-Solomon code for Facebook).
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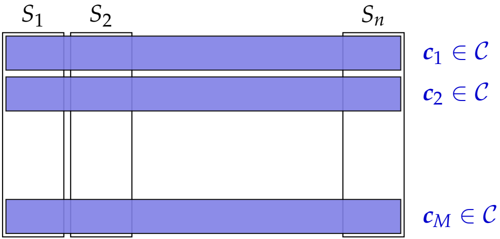
Given a code $\mathcal{C}$ of length n:



Definition (Reed-Solomon code). Let $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{F}_q^n$, pairwise distinct.

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$\mathcal{C} = \mathrm{RS}_q(k,n)$ is **MDS**:

- every codeword $\boldsymbol{c} \in \mathcal{C}$ can be reconstructed from any k-subset of coordinates of $\boldsymbol{c}$,
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File storage:

$$\text{a file} \quad F_i \in \Sigma \simeq \mathbb{F}_{q^s}^k \quad \text{is encoded into} \quad \boldsymbol{c}_i \in \mathrm{RS}_q(k,n) \otimes \mathbb{F}_{q^s}$$

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Main assumption (can be discussed):

$$s \gg M$$

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Presentation

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Next, we present a PIR scheme for RS-coded databases.

- Originally [TR16], then extended and reformulated [TGKFH18, TGR18].
- Scalable.
- Optimal PIR rate for $t = 1$ and $M \to \infty$.
- PIR rate conjectured optimal for $M \to \infty$.

[TR16] *PIR from MDS Coded Data in Distributed Storage Systems*. Tajeddine, El Rouayheb. ISIT. **2016**.

[TGKFH18] *Robust PIR from Coded Systems with Byzantine and Colluding Servers*. Tajeddine, Gnilke, Karpuk, Freij-Hollanti, Hollanti. ISIT. **2018**.

[TGR18] *PIR from MDS Coded Data in Distributed Storage Systems*. Tajeddine, Gnilke, El Rouayheb. IEEE-TIT. **2018**.

The protocol: query generation

Notation:
$$\boldsymbol{a} \star \boldsymbol{b} := (a_1 b_1, \dots, a_n b_n)$$
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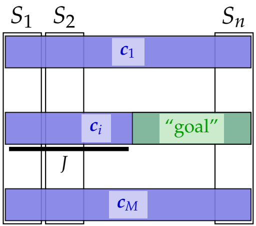
System parameters:

$\mathcal{C} \subseteq \mathbb{F}_q^n$ the *storage code*, $\boldsymbol{C} \in \mathcal{C}^M$ the coded database

$\mathcal{D} \subseteq \mathbb{F}_q^n$ a *query code* of dual distance $d^\perp(\mathcal{D}) = t + 1$

$J \subseteq [1, n]$ an information set for $\mathcal{C} \star \mathcal{D}$, and $\bar{J} := [1, n] \setminus J$



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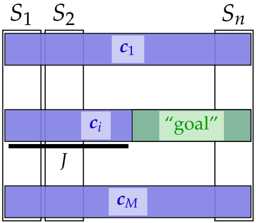

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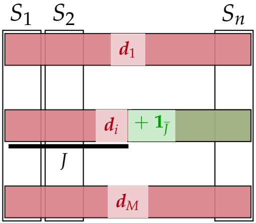
Queries:

1. the user generates at random M words $\boldsymbol{d}_1, \dots, \boldsymbol{d}_M \in \mathcal{D}$ and defines $\boldsymbol{Q}$ as follows:
2. the j-th column of $\boldsymbol{Q}$ is sent to server S_j



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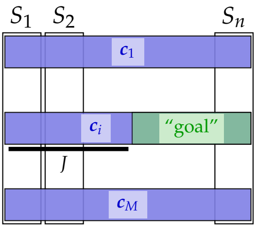
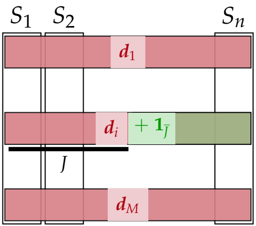


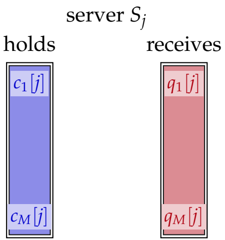
Queries:

1. the user generates at random M words $\boldsymbol{d}_1, \dots, \boldsymbol{d}_M \in \mathcal{D}$ and defines $\boldsymbol{Q}$ as follows:
2. the j-th column of $\boldsymbol{Q}$ is sent to server S_j



Remark: queries remain private against collusions of servers of size $\leq t$.

Server answers: server S_j receives as a query a column $\boldsymbol{Q}^{(j)} \in \mathbb{F}_q^M$ of $\boldsymbol{Q}$,



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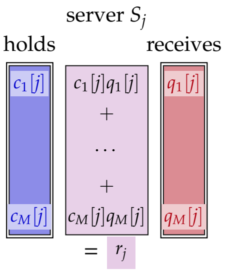

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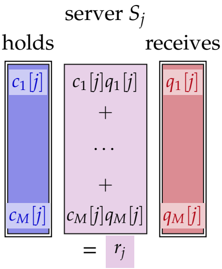

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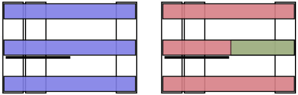

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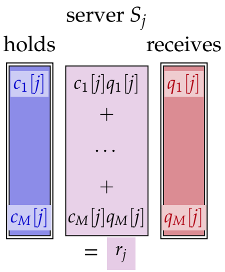
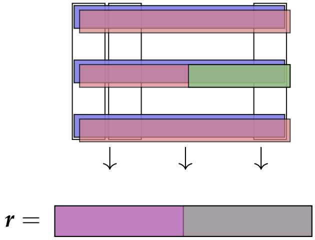


Reconstruction: The user collects

$$\boldsymbol{r} = (r_1, \dots, r_n) = \underbrace{\sum_{m=1}^{M} \boldsymbol{d}_m \star \boldsymbol{c}_m}_{\in \mathcal{C} \star \mathcal{D}} + \underbrace{\boldsymbol{1}_{\bar{J}} \star \boldsymbol{c}_i}_{= \boldsymbol{c}_i \text{ on } \bar{J}}$$



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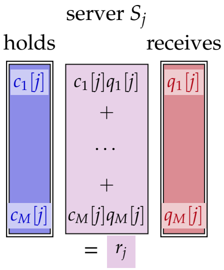

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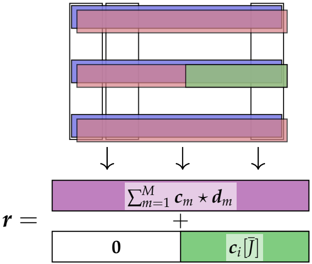
and interpolates on J to recover

- $\sum_{m=1}^{M} \boldsymbol{d}_m \star \boldsymbol{c}_m$,
- then $\boldsymbol{c}_i[|\bar{J}]$.



Features for 1 run of the protocol.

- download cost: n symbols over $\mathbb{F}_{q^s}$
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Otherwise, **striping** methods allow to achieve the same PIR rate.

Outline

1. Private information retrieval

2. PIR schemes for common storage systems
   Distributed storage systems
   A PIR scheme on RS-coded databases
   A PIR scheme with regenerating codes

3. PIR schemes with low computation
   Transversal designs and codes
   A PIR scheme with transversal designs
   Instances

4. Conclusion

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Definition: $\mathcal{C}$ is an $(n, k, d, \alpha, \beta, B)$-**regenerating code** if:

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A particular optimal point (minimum-bandwidth repair, MBR): $d\beta = \alpha$.
Then,

$$B = \left(kd - \frac{k(k-1)}{2}\right)\beta\,.$$

Product-matrix MBR codes [RSK11]

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$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{S} & \boldsymbol{T}^\top \\ \boldsymbol{T} & \boldsymbol{0} \end{pmatrix}$$

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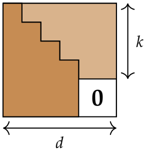

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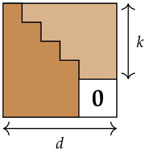
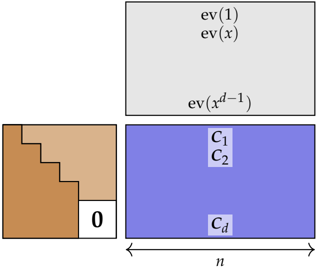


2. Let G be a $(d \times n)$ generator matrix for $\mathrm{RS}_q(d, n)$, echelonized in degree (*i.e.* a Vandermonde matrix). Codewords are then:

$$C = AG \in \mathbb{F}_q^{d \times n} .$$



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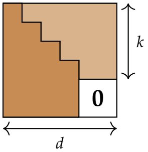

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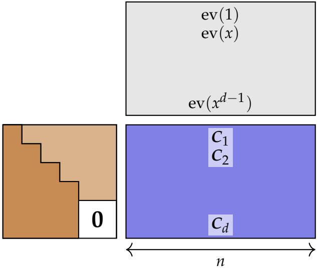

Remark: row $\boldsymbol{C}_j$ of $\boldsymbol{C}$ is a word of a RS code

- of dimension k, if $j > k$,
- of dimension $d > k$ otherwise.

PIR scheme on PM-MBR codes with no collusion

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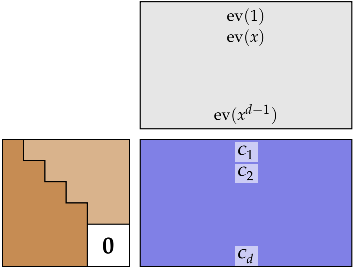

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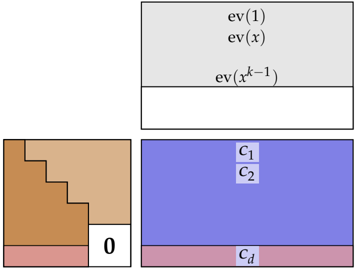

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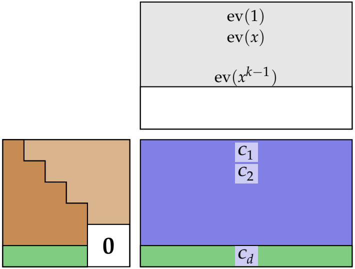

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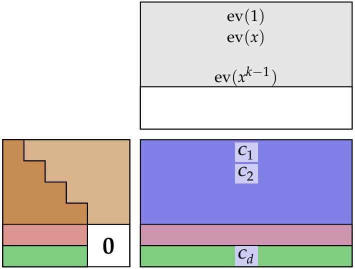

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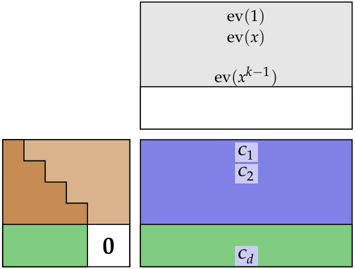

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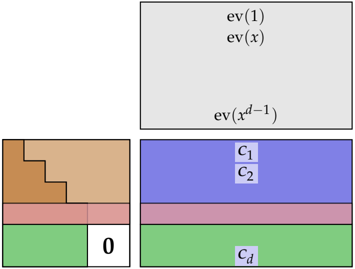

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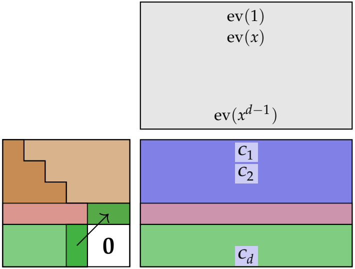

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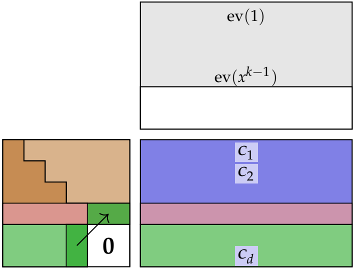

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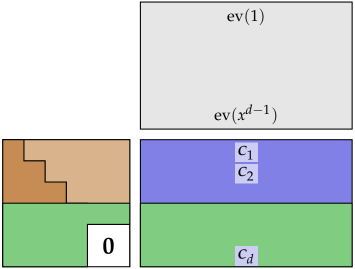

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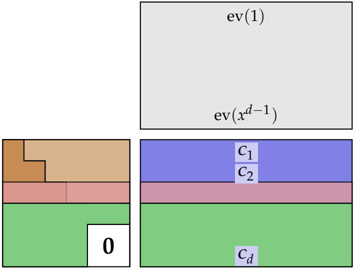

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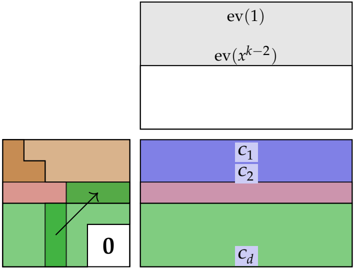

PIR scheme on PM-MBR codes with no collusion

Private Information Retrieval Schemes with Regenerating Codes. L., Tajeddine, Freij-Hollanti, Hollanti. `arxiv:1811.02898`. **2018**.

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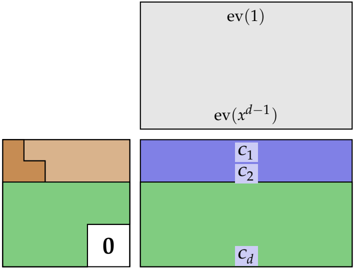

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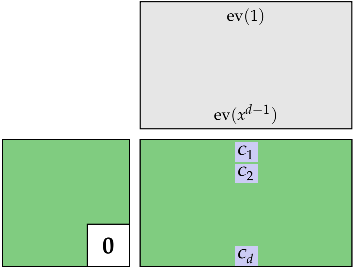

Retrieval rate: $1 - \frac{j}{n}$

We get a PIR rate:

$$\rho = \frac{1 - \frac{k}{n}}{1 - \frac{k(k+1)(k-1)}{nB}} \; > \; 1 - \frac{k}{n}$$

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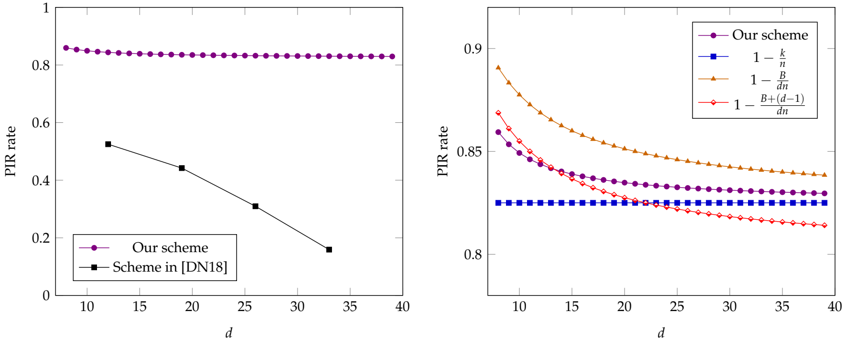

Comparison of PIR rates for $n = 40$ and $k = 7$.

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Previous schemes:

- low communication complexity
- computationally inefficient (linear in $|F| = \sum_{m=1}^{M} |F_m|$)

Our goal:

- optimal computation ($|r_j|$ for each server S_j)
- remove the assumption $s \gg M$
- moderate communication complexity

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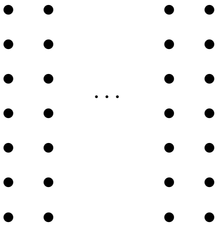
Transversal designs

A **transversal design** $\mathrm{TD}(n, s) = (X, \mathcal{B}, \mathcal{G})$ is given by:

- X a set of *points*, $|X| = N = ns$,



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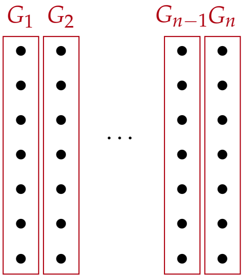

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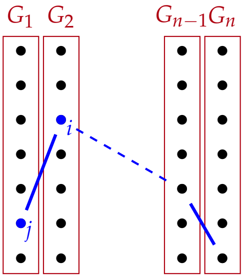
- *blocks* $B \in \mathcal{B}$ satisfying
  - $B \subset X$ and $|B| = n$;
  - for all $\{i,j\} \subset X$, $\{i,j\}$ lie:
    either in a single group $G \in \mathcal{G}$,
    or in a unique block $B \in \mathcal{B}$



Let $\mathcal{T}$ be a transversal design $\mathrm{TD}(n, s) = (X, \mathcal{B}, \mathcal{G})$.

Its **incidence matrix** $\boldsymbol{M}$ has size $|\mathcal{B}| \times |X|$ and is defined by:

$$M_{i,j} = \begin{cases} 1 & \text{if } x_j \in B_i \\ 0 & \text{otherwise.} \end{cases}$$

Codes from designs

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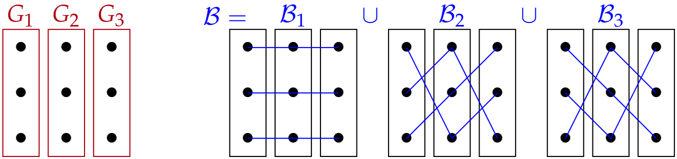
The **code** $\mathcal{C}$ **based on** $\mathcal{T}$ **over** $\mathbb{F}_q$ is the $\mathbb{F}_q$-linear code admitting $\boldsymbol{M}$ as a parity-check matrix ($\mathcal{C}^\perp$ is generated by $\boldsymbol{M}$).

- $\mathrm{length}(\mathcal{C}) = |X|$,
- $\dim(\mathcal{C}) = \dim(\ker \boldsymbol{M})$,
- every $B \in \mathcal{B}$ gives an $\boldsymbol{h} \in \mathcal{C}^\perp$ such that $\mathrm{wt}(\boldsymbol{h}_{|G_j}) = 1$, $\forall j = 1, \dots, n$.

Example

The transversal design $\mathrm{TD}(3,3)$ represented by:



gives an incidence matrix

$$\boldsymbol{M} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Its rank over $\mathbb{F}_3$ is 6 $\quad\Longrightarrow\quad$ the associated code $\mathcal{C}$ is a $[9,3]_3$ code.

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Let $\mathcal{C} \subseteq \mathbb{F}_q^N$ be a code based on a $\mathrm{TD}(n, s)$.

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- **Initialisation.** User U encodes $F \mapsto \boldsymbol{c} \in \mathcal{C}$, and gives $\boldsymbol{c}_{|G_j}$ to server S_j.
- **To recover** $F_i = c_i$, with $i \in X$:
  1. User U randomly picks a block $B \in \mathcal{B}$ containing i.
     Then U defines:

$$q_j = \mathcal{Q}(i)_j := \begin{cases} \text{unique } \in B \cap G_j & \text{if } i \notin G_j \\ \text{a random point in } G_j & \text{otherwise.} \end{cases}$$

  2. Each server S_j sends back c_{q_j}
  3. U recovers

$$c_i = -\sum_{j:\, i \notin G_j} c_{q_j} = -\sum_{b \in B \setminus \{i\}} c_b$$

Theorem. This PIR protocol is information-theoretically private.

Proof:

- the only server which holds F_i received a random query;
- for each other server S_j, query q_j gives no information on the block B which has been picked $\Rightarrow$ no information leaks on i.

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- communication complexity: $n \log s$ uploaded bits, $n \log q$ downloaded bits
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  - **only** 1 **read for each server** (somewhat optimal)
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Question: transversal designs with good $\dim(\mathcal{C})$ depending on (n, s)?

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$\mathcal{T}_A$, the **classical affine transversal design**:

- $X = \mathbb{F}_q^m$, $m \geq 2$,
- $\mathcal{G}$ a set of q disjoint hyperplanes partitionning X,
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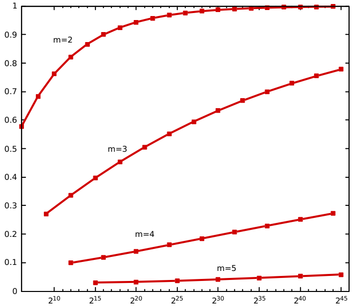

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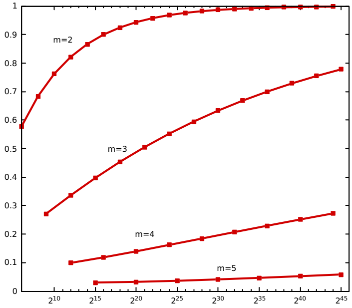

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Question: better instances?

An **orthogonal array** $\mathrm{OA}(t, n, s)$ of strength t is a list A of words
- of length n,
- over a finite set S, $|S| = s$,
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Equivalently, an $\mathrm{OA}(t, n, s)$ is a code $A \subset S^n$ with dual distance $t + 1$.

$$S = \{a, b\}$$

$$\mathrm{OA}(2,3,2) = \begin{bmatrix} a & b & b \\ b & b & a \\ b & a & b \\ a & a & a \end{bmatrix}$$

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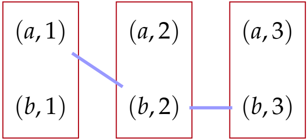

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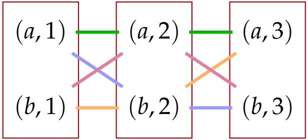

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Proposition. For $t = 2$, an $\mathrm{OA}(t, n, s)$ gives a $\mathrm{TD}(n, s)$.

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For $t \geq 3$, we get TDs such that:

for every t-set T of points lying in t different groups,
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$\Rightarrow$ The PIR protocol resists $t - 1$ colluding servers.

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- OAs with $t > 2$ exist (*e.g.* from Reed-Solomon codes)
- But associated TDs lead to codes with poor rates (except for $t \ll n$)

Private Information Retrieval from Transversal Designs. L.. IEEE-TIT. **2019**.

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